B Alexander

NUMERICAL PROCEDURES IN THE ANALYSIS OF INDETERMINATE STRUCTURES

The following is presented to illustrate the use of Newmark's numerical procedures in the analysis of indeterminate structures. The source of material for this portion of the report is the notes obtained from lectures by Dr. R.N. McManus.

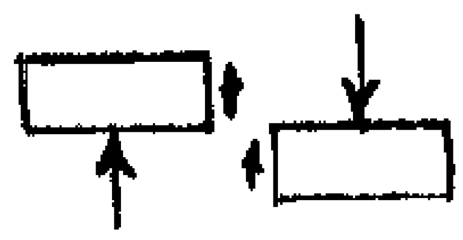
Numerical procedures are a system of bookkeeping and an application of basic assumptions and basic principles to obtain the solution to complex problems in engineering. Numerical procedures are used to best advantage in the solution of difficult problems which require a great deal of time for an exact solution. In the use of numerical procedures, the computations can be arranged in tabular form and after this has been done correctly, the solution is readily obtained. In this way no time is wasted. The results are always almost identical with those obtained from highly theoretical solutions and the amount of variation is negligible. Thus, these procedures can be termed short cut methods to the solution.

It is assumed that the reader is thoroughly acquainted with the theory studied in U. of A. Civil Engineering courses C.E. 11, C.E. 60 and C.E. 61.

1. Short-cut Methods

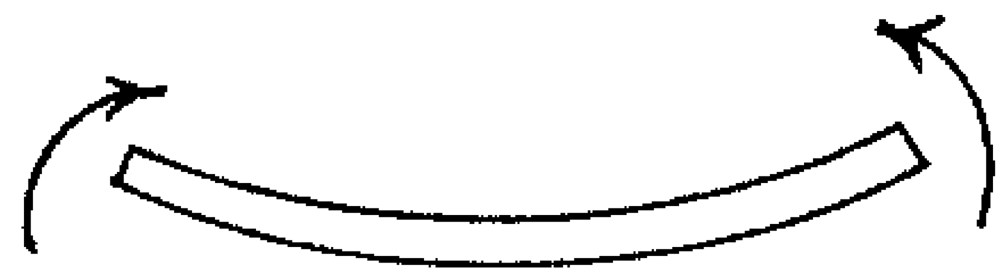
(1-1) Sign Convention - to begin with, as in all methods studied so far, convention is necessary. The sign convention to be used from here on will be as follows: LOADS: positive loads act vertically upward (i.e. $\uparrow$ (+ve))
SHEARS: positive shears occur when the left part of the beam, etc. tend to move upward with respect to the right part.

That is, for positive shear, the section must tend to arrange itself thus -



Also, there is a net clockwise couple set up about an axis in the section in question due to the external forces acting.

MOMENT: positive bending moment occurs when the beam bends so that it is concave upward. The shape of the deformed beam is as shown here for +ve bending moment.



That is, for beams in pure bending, the end moment at the left end is clockwise in direction and that at the right end is counterclockwise.

(1-2) Fundamental Procedure
Suppose we have a simple beam (Fig. 1-1) loaded as shown. The reactions are: $R_L = R_R = P/2$. The resulting shear force diagram can now be drawn. See Fig. 1-1(a).
From the principle that the change in BM between 2 sections is equal to the area of the SF diag. between the sections, the BM diag. can be drawn. (Fig. 1-1(b)).

Now suppose that an error had been made in computing the reactions and that they were found to be $R_L = 3/8\ P$
$R_R = 5/8\ P$

for the beam shown. The resulting SF diag and BM diag are shown in parts (c) & (d) of Fig. 1-1. Since this is a simply supported beam, the bending moment at either end must be equal to zero. Therefore the bending moment, at the left end is correct and that at the right end is in error by $-\frac{2P\ell}{16}$.

Now the error made in computing the reactions produced a constant error in shear force and since the bending moment varies linearly as the length of the shear force diagram, a "linear" change is produced in the error in bending moment.

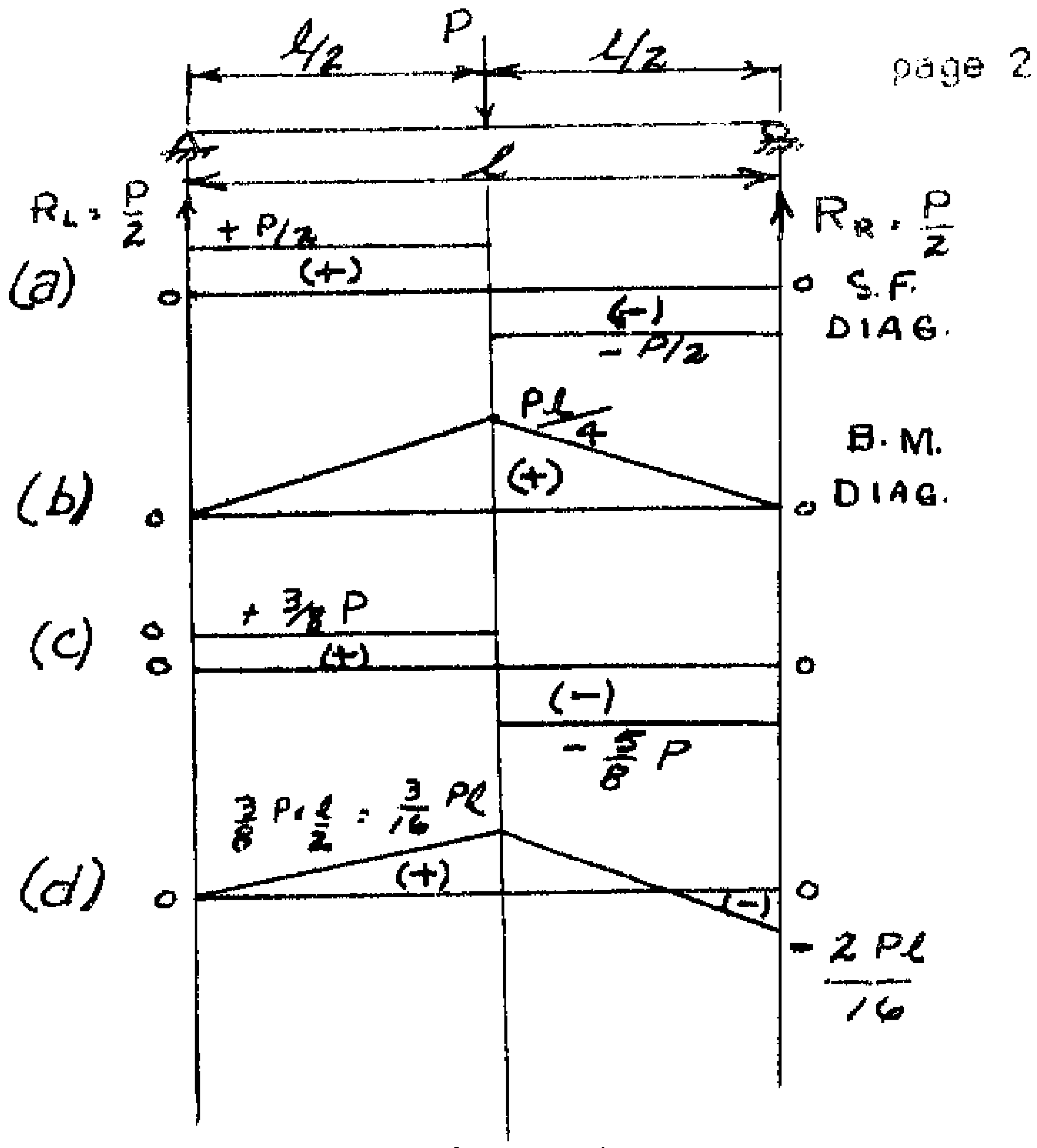


Fig 1-1

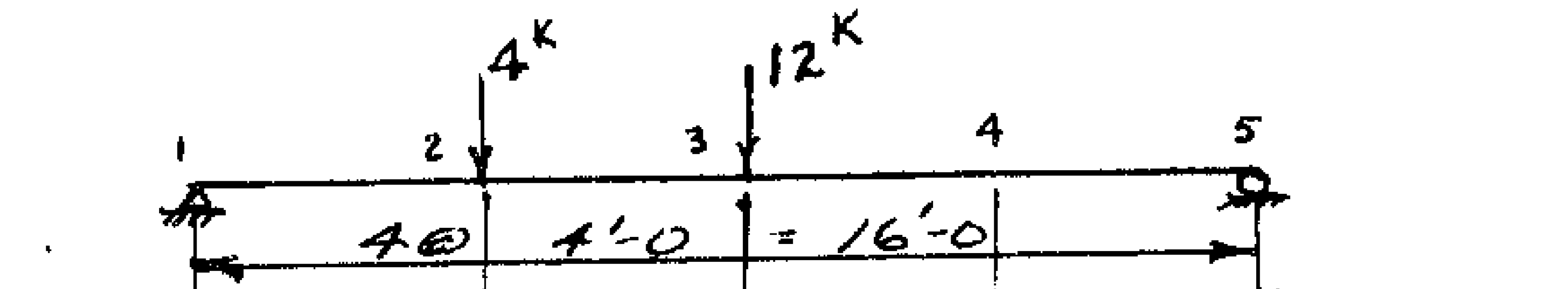
Now, to produce the correct BM diag suppose we rotate the BM diag (Fig. 1-1(d)) counter-clockwise so that the BM is equal to zero at the right end of the beam. Therefore, in a sense we have added a linearly varying correction moment varying over the entire length from zero at the left end to $+\frac{2P\ell}{16}$ at the right end. That is, a change of $+\frac{2P\ell}{16}$ at the right end produces a change of $+\frac{1P\ell}{16}$ at the center of the beam. A change of $+\frac{P\ell}{16}$ in BM gives a change of $+\frac{P\ell}{16} / \frac{\ell}{2} = +P/8$ in the shear force, which, when applied to the shear force in Fig. 1-1(c) gives the correct value of shear force.

Example (1)

We have a simple beam divided into 4 equal panels and loaded with a series of loads. It is required to compute the shears and bending moments at the panel points. The fundamental procedure is to start out with a bookkeeping system. The reactions are not required for this process.

Let $R_L = x_1$
$R_R = x_2$

Then shear at panel point 1 = x_1 and shear at panel point 5 = x_2. Assume some value for x_1. Then the shear force across panel 1-2 will be equal to the value assumed for x_1.



	1		2		3		4		5	
P	x_1		-4		-12		0		x_2	Kips
V		+6		+2		-10		-10		Kips
m'	0		+6		+8		-2		-12	$\times 4^{1K}$
Mc	0		+3		+6		+9		+12	$\times 4^{1K}$
MF	0		+9		+14		+7		0	$\times 4^{1K}$
VF		+9		+5		-7		-7		Kips

Fig. 1-2

Let x_1 = +6 kips. Then shear force across panel 1-2 is +6 kips. At panel point 2, the shear force changes suddenly to $+2^k$ and since there are no further loads coming onto panel 2-3, the shear force remains constant across that panel. Similarly the shear force across panels 3-4 and 4-5 is -10^k.

The vertical loads are arranged in row P in Fig. 1-2. The shears are in the row marked V.

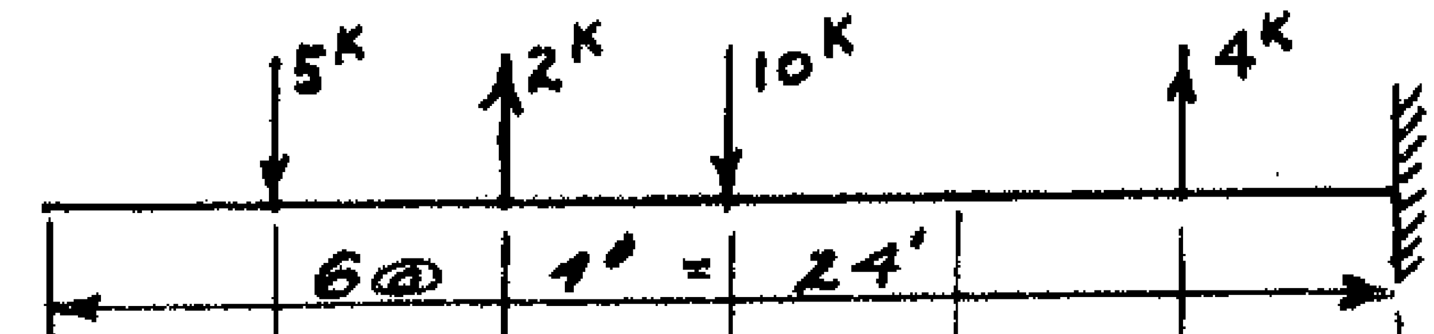
Now, since change in BM between two sections equals the area of the SF diag between the two sections, we can compute the bending moments M' at the panel point on the basis of our previously assumed value for shear force. Since the panels are all of equal length, the factor 4 ft kips can be removed and placed in the right most column. See Fig. 1-2.

Apply a linear correction moment M_c, and the sum of M' and M_c gives the final correct moment M_f at each panel point. Then the correct shear force is determined.

Finally $R_L = +9^k$ and $R_R = +7^k$ and $\Sigma V = 0$.

<u>Example (2)</u>
Suppose we have a cantilever beam loaded as shown in Fig. 1-3. It is required to compute the shears and bending moments across each panel.

For a cantilever beam no correction moments are required because the shear is known at the free end of the beam.



P	0	-5	+2	-10	0	+4		kips
V	0	-5	-3	-13	-13	-9		Kips
M	0	0	-5	-8	-21	-34	-43	× 4'k
MF	0	0	-20	-32	-84	-136	-172	ft·kips

Fig. 1-3

(1-3) <u>Equivalent Concentrations.</u>
We have shown how the fundamental procedure has been applied to simple beams with a single concentrated force. We will now proceed to show the fundamental procedures to be used when dealing with uniform loads. To do this it is necessary to introduce the idea of equivalent concentrations. We will proceed to show that with the use of equivalent concentrations, the BM curve obtained is the correct one and agrees with the original whereas the SF curve usually is not correct and that a linear correction can be applied to give the correct SF curve.

To begin with, let us consider a simple uniformly loaded beam of span length ℓ and loading equal to w#/ft as shown in Fig. 1-4(a). Now suppose that we load the original beam with two simple beams of span $\ell/2$ loaded uniformly to w#/ft as in Fig. 1-4(b). In effect, we have in this case a concentrated load of $\frac{w\ell}{4}$ at each end of the original beam and two concentrated loads of $\frac{w\ell}{4}$ or one concentrated load of $\frac{w\ell}{2}$ at the center. The reactions still remain $\frac{w\ell}{2}$ at each end. Plotting the SF diagram, we see that we do not get the correct shear force curve. Fig. 1-4(c). To obtain the correct shear, a correction of $\pm\frac{w\ell}{4}$ must be applied at the point of action of the effective concentration. That is, adding $+w\ell/4$ to the shear at A gives the correct value of shear = $w\ell/4 + w\ell/4 = w\ell/2$. Similarly a correction of $-w\ell/4$ @ the right end of the first trolley gives zero for the shear at the centerline which is the correct value. Applying a correction $+w\ell/4$ to the shear at the left end of the second trolley also gives zero for the shear at the centerline. At the right end of the beam, a correction of $-w\ell/4$ applied to the right end of the second trolley gives a value $-w\ell/2$ for the shear at that section. It can be shown that the curve obtained in Fig. 1-4(c) gives the correct value for the shear at the point $\ell/4$, $\ell/2$ and $3\,\ell/4$ of the span of the original beam.

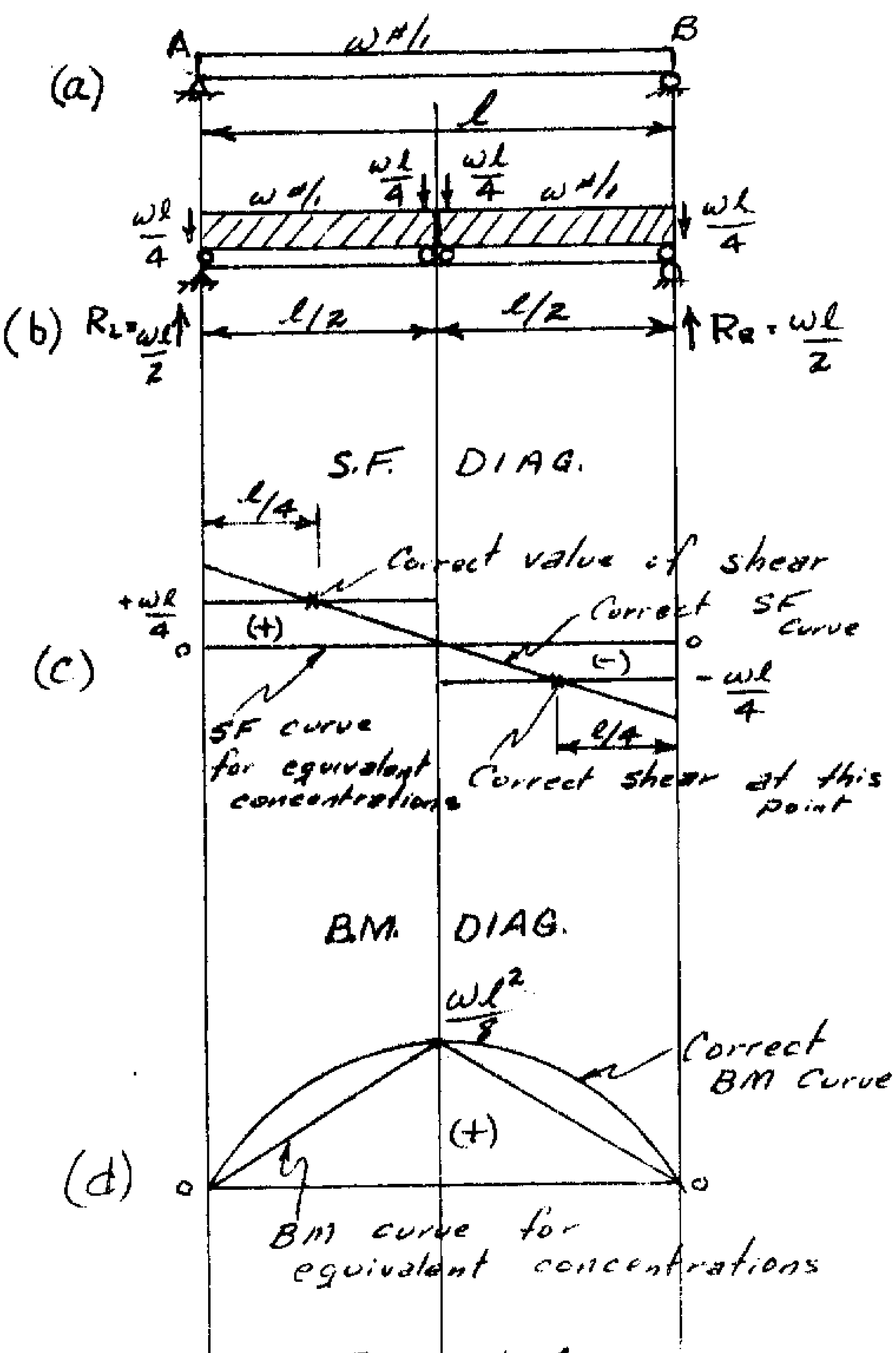


Fig. 1-4

Now to obtain the correct value of BM at the centerline either SF curve may be used. That is the BM at the ℄ = $w\ell/4 \times \ell/2 = w\ell^2/8$ which is the correct value of BM at the ℄.

To obtain a closer approximation to the correct BM curve, the beam can be made to support as many trolleys as desired. For example, if we use four trolleys we will have - (Fig. 1-5).

The curve obtained for the BM curve approaches a parabola. Taking the origin at the curve at the vertex and going a distance of 1/2 the way to the end of the beam the decrease is $(1/2)^2 = 1/4$ which corresponds to the point required. That is, we always obtain the correct value for BM at the point of the concentration.

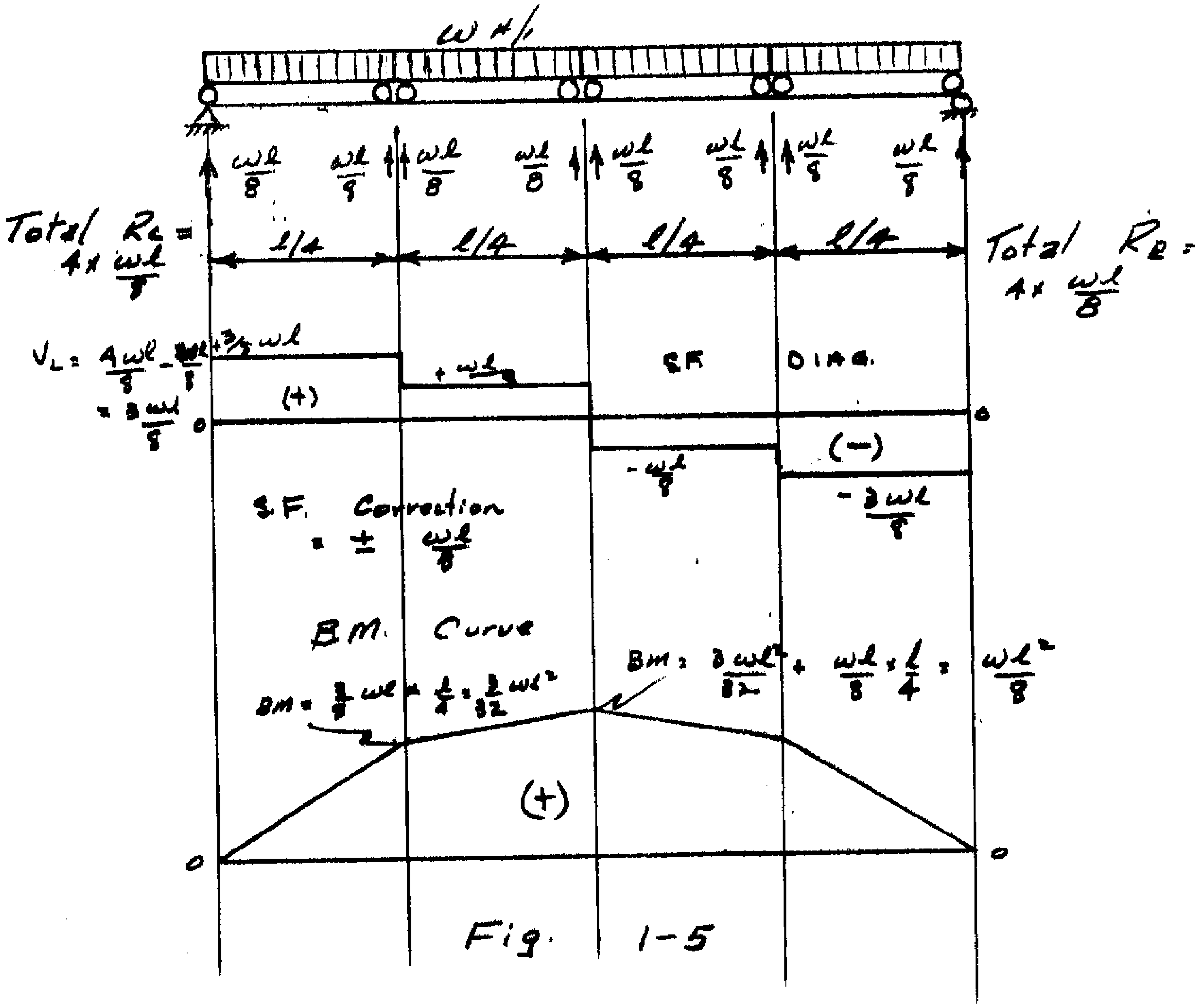
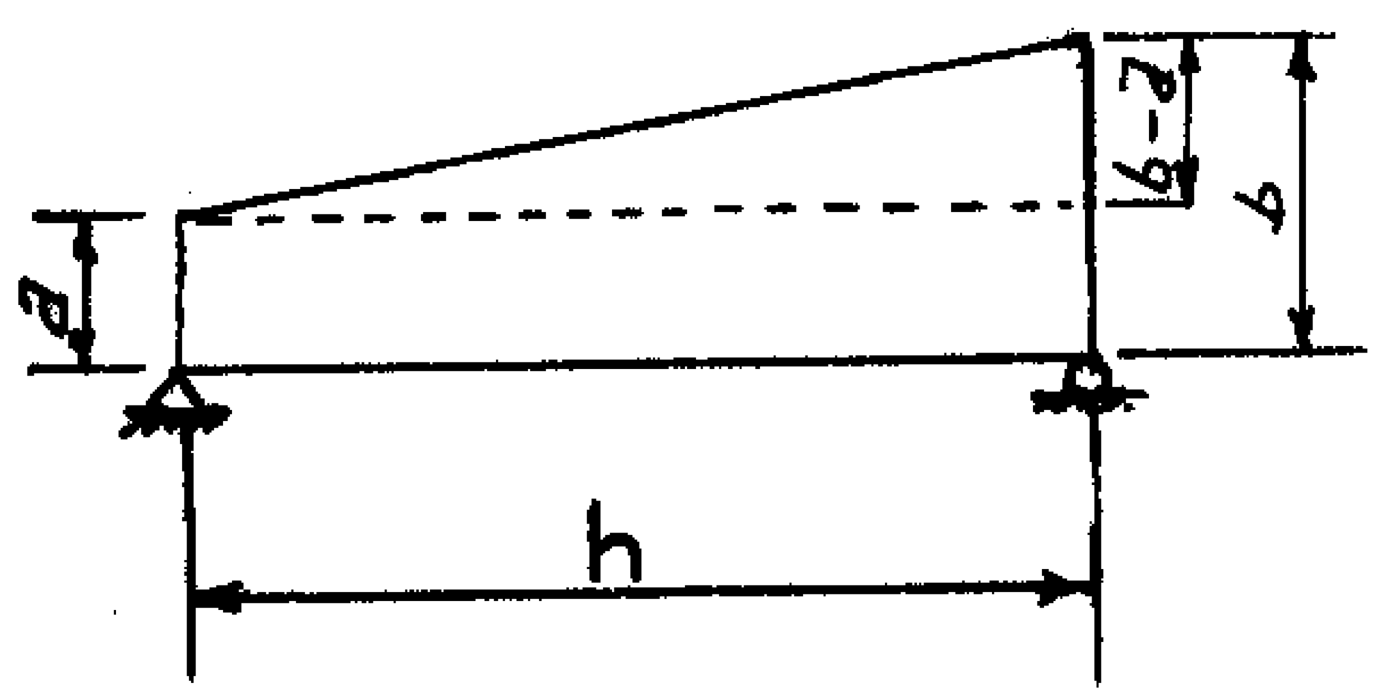


Fig. 1-5

(1-4) Trapezoidal Loadings.

The following deals with effective concentrations due to trapezoidal loadings. Suppose we have the following -



h = increment of span

Fig 1- 6

$$R_L = a\frac{h}{2} + \frac{1}{3}(b-a)\frac{h}{2}$$

$$= \frac{3ah + bh - ah}{6}$$

$$= \frac{h}{6}(2a + b)$$

$$R_R = a\frac{h}{2} + \frac{2}{3}(b - a)\frac{h}{2}$$

$$= \frac{3ah + 2bh - 2ah}{6}$$

$$= \frac{h}{6}(2b + a)$$

Always, the reaction for a trapezoidal loading is given by multiplying the sum of twice the value of the ordinate at the point and the value of the ordinate at the other end of the loading curve by one-sixth the height. Combining the ordinates of 2 adjacent increments, we have as shown below -

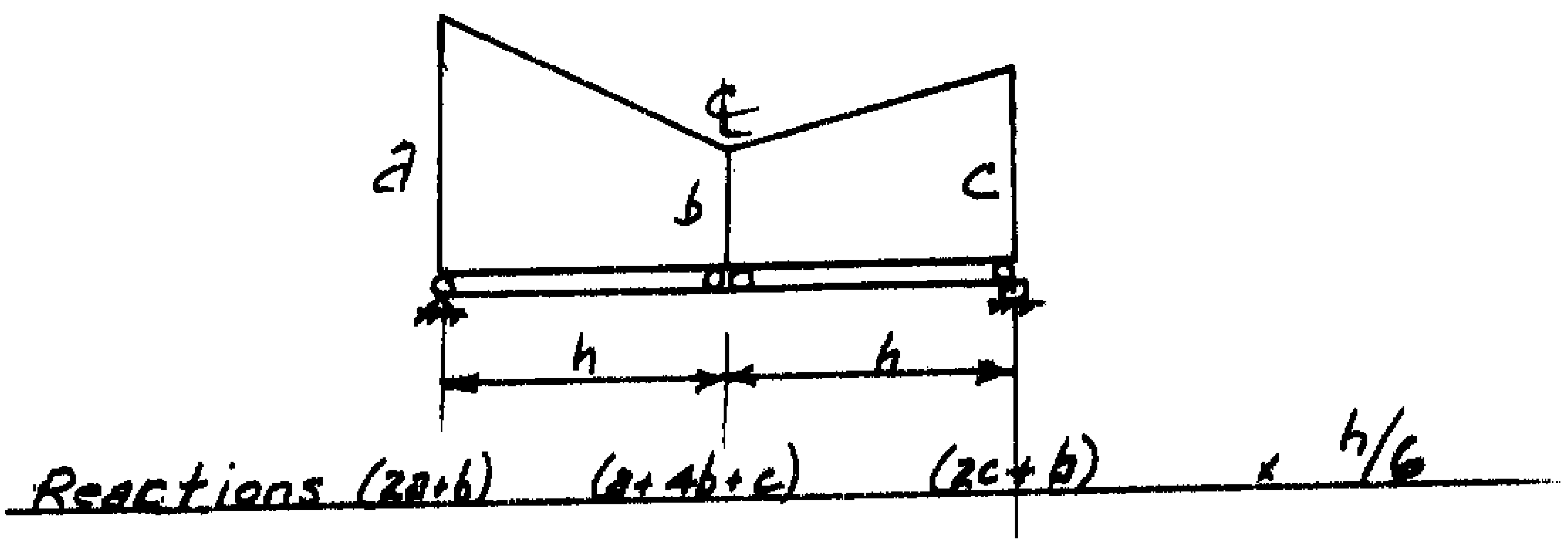


Fig 1-7

(1-5) <u>Non-uniform, non-trapezoidal loadings.</u>
If we have a very irregular loading, a second degree parabola as approximated by Simpson's Rule, will be used in increments to obtain the approximate loading curve as shown in Fig. 1-8, and if we know the values of the ordinates at each of the points, an expression for the equivalent concentration can be derived. These expressions are always as given in Fig. 1-9.

Fig. 1-8

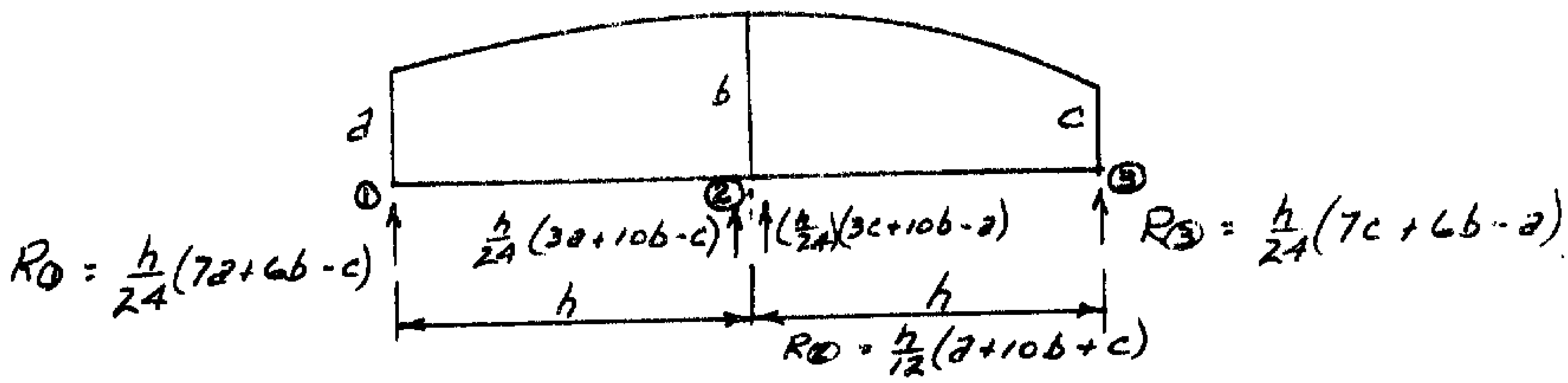


Fig 1-9

We are not always too concerned with the end concentrations. That is, we can assume a value for shear in the end panel and apply a linear correction.

(1-6) <u>Deflections</u>

(1) Geometry of small angles.

Here we apply a limitation to the angle change that can occur. Therefore we are always within the allowable error. That is, we deal with angles for which correct to three significant figures the following holds -

$$\sin \alpha = \tan \alpha = \alpha$$

That is, we will deal with angles generally less than 5°. Therefore, all movements due to rotation will be treated as being perpendicular to the radius that is rotated. Suppose we have a rigid body as shown in Fig. 1-10. Suppose we rotate it through the angle α.

$$\Delta = R\,\alpha$$

$$\Delta_V = R\,\alpha \cos\beta$$

$$\Delta_V = (R\cos\beta) \times \alpha$$

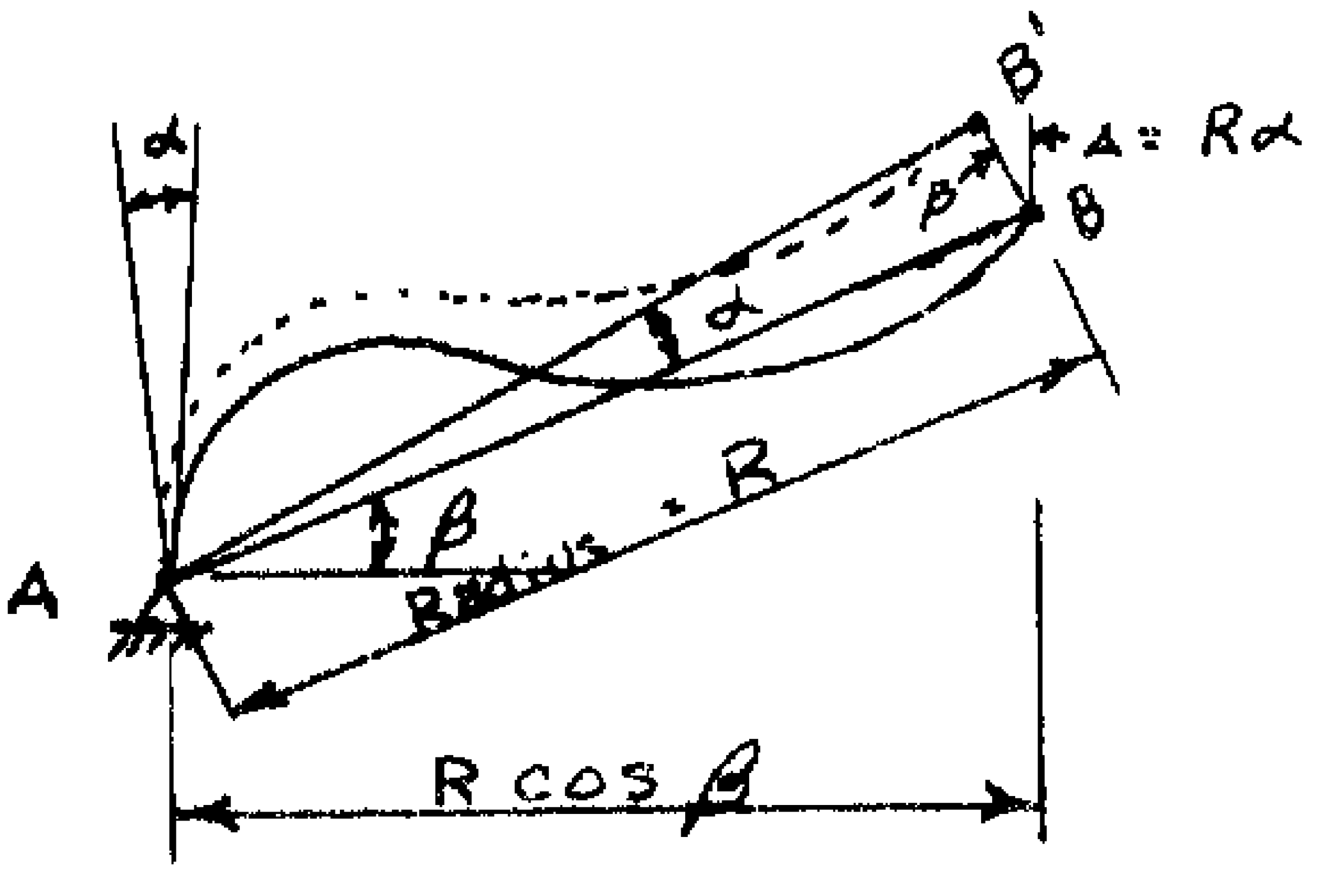


Fig. 1-10

Therefore if we know the point of rotation, the movement or deflection is equal to the radius to point perpendicular to direction of movement multiplied by the angle change. The only assumption to be used for above is that the angle change is small.

From this as a basis, the geometry of lines will be developed

(ii) Geometry of lines. Suppose we have a straight line AB divided into segments as shown and that at each point there is a corresponding angle change as the line deflects to the positions A, C', D', E', B.

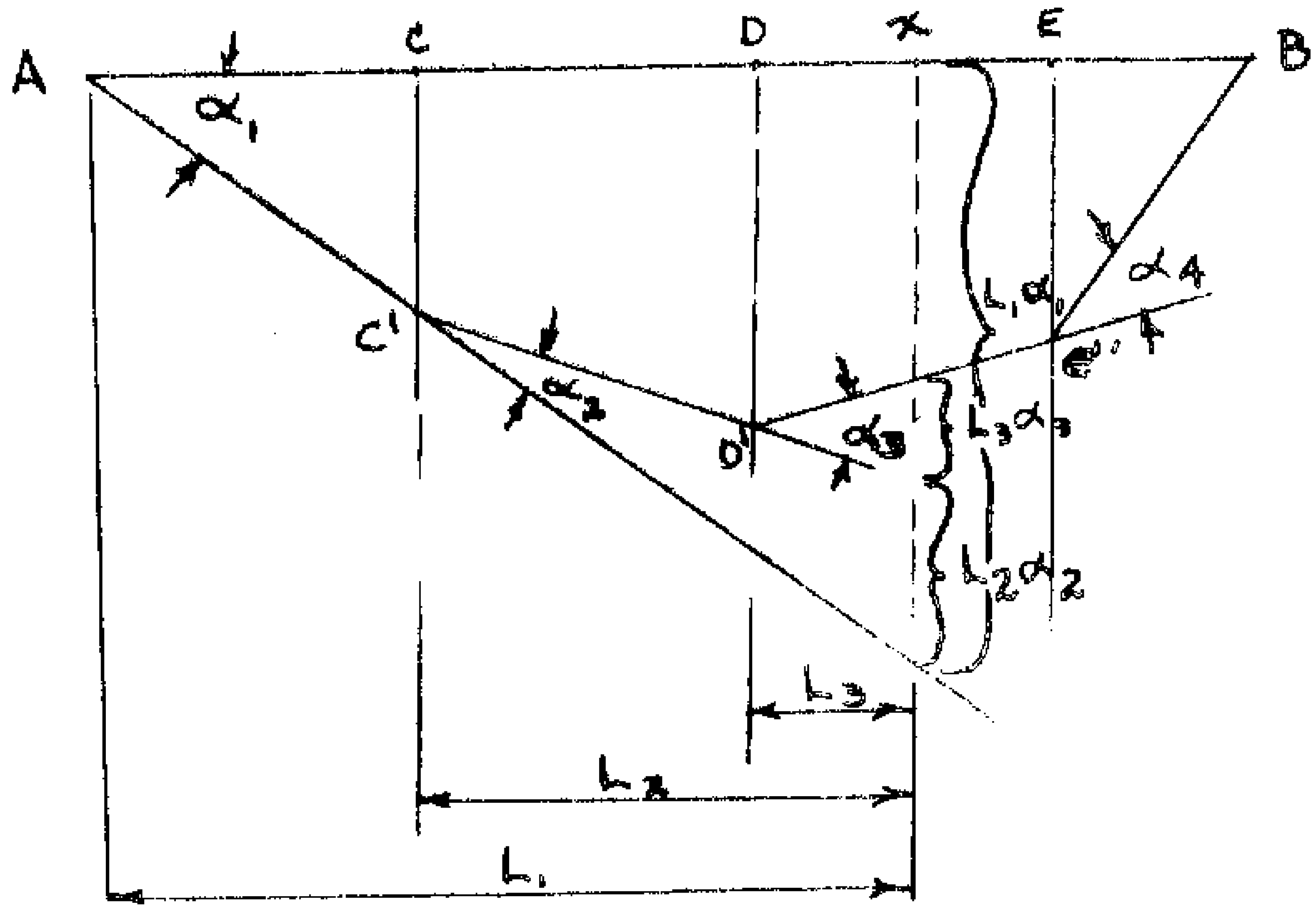


Fig 1-11

Then the departure of the line at x is equal to the deflection of the line from its original position. i.e. the departure = $L_1\alpha_1 - (L_2\alpha_2 + L_3\alpha_3)$

Also $L_1\alpha_1$ = statical moment of α_1 about x. Similarly $L_2\alpha_2$ = statical moment of α_2 about x and $L_3\alpha_3$ = statical moment of α_3 about x. The moment arm is measured perpendicular to the direction of movement.

That is, the departure is equal to the algebraic sum of the statical moments of all the angle changes to the left of the section about an axis in the section. This is analogous to the shear force and bending moment at the section. The shear and BM can be converted by use of the elastic properties to give the slope and deflection respectively of a loaded beam.

(1-7) Angle Changes.

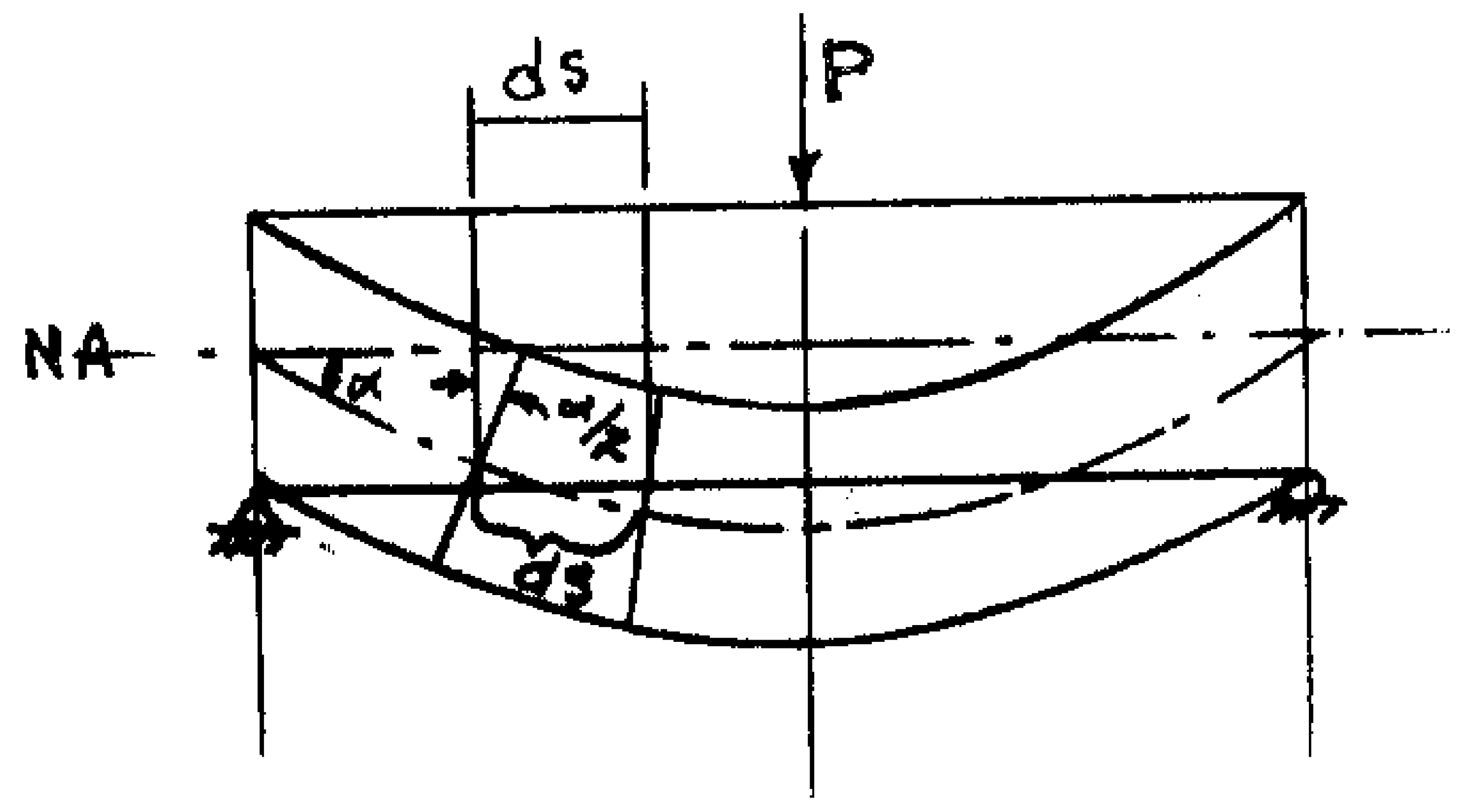


Fig. 1-12

Consider a simple beam loaded as shown in Figure 1-12. When load is applied to a beam the beam bends and deflects. That is sections rotate about an axis through the N.A., and the element of length ds remains unchanged at the neutral axis whereas the fibre in the compression zone shortens and that in the tension zone lengthens. As a result there are produced between the sections, angle changes due to this shortening on the one side and the lengthening on the other. We can assume that each section rotates through half the angle change which is small to begin with.

Now the fibre stress, $f = My/I$
where f = fibre stress,
M = moment at the section,
y = distance from the NA to the fibre in question and
I = moment of inertia of the section about an axis through the NA.

The unit strain, $e = f/E$
where E = mod. of elasticity.

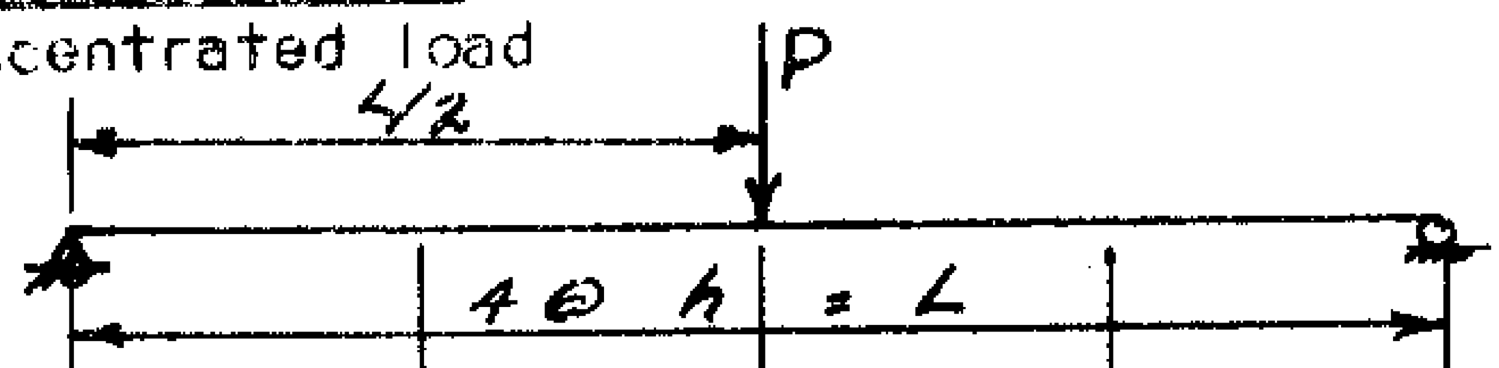
Therefore the total strain over the length of the element $ds = e\,ds$
$= f/E \,.\, ds$.

If we deal with small angles, $d\alpha = 2 \times \frac{dx}{2} = \frac{e\,ds}{y}$. That is, we are using the basic assumption that $\alpha = \tan\alpha = \frac{eds}{y}$
measured in angular measure in radians.

Also, $d\alpha = f/Ey \,.\, ds$ which in turn is equal to $My/EIy \,.\, ds = M/EI \,.\, ds$.
If we let ds = one inch, we obtain $d\alpha = M/EI$ radians/inch.

(1-8) Procedures for Finding Deflections.

(1) Simple beam - concentrated load



Due to symmetry, P & V are known at ℄. Therefore, begin P & V rows from the center

$\frac{h}{6}(a+4b+c)$

P	x		0		-1		0		x	P^K
V		$+\frac{1}{2}$		$+\frac{1}{2}$		$-\frac{1}{2}$		$-\frac{1}{2}$		P^K
M	0		+1		+2		+1		0	$\frac{Ph}{2}$ IK
α	0		-1		-2		-1		0	$\frac{Ph}{2EI}$ $\frac{rad}{ft}$
$\bar{\alpha}$	x		-6		-10		-6		x	$\frac{h}{6}\,\frac{Ph}{2EI}$ rad
ϕ		+11		+5		-5		-11		$\frac{h}{6}\,\frac{Ph}{2EI}$ rad.
y	0		+11		+16		+11		0	$\frac{Ph^3}{12EI}$ ft.

Fig. 1-13

Sign Convention

1. Angle change obtained from positive moment is considered negative. α (-)
2. Deflection downward is considered positive. (+)
3. Downward slope is positive. ϕ (+)
4. Downward load is negative. (-)
5. Standard moment and shear conventions are used.

* Procedure

P - End reactions not considered.
V - Due to symmetry, correct shear is known.
M - Correct moment.
α - Angle change at panel points in relation to BM = M/EI (see sign convention).
$\bar{\alpha}$ - Beam is broken into equivalent concentrations. Let $\bar{\alpha}$ be the equivalent angle change concentration. Trapezoidal loading is used.
ϕ - Slope of panel.
y - Deflection of beam at panel points.

Deflection at ℄

$y_{℄} = 16\ Ph^3/12EI = 4/3\ Ph^3/EI$ but $L = 4h$

therefore $y_{℄} = PL^3/48EI$ which is the correct value.

Deflection at 1/4 point

$y_{1/4} = 11\ Ph^3/12EI = 11\ PL^3/768\ EI$

(II) Simple beam - Uniform load - Use of equivalent concentrations

P		-1	-1	-1	-1	-1		w
$\bar{P}$		-6	-6	-6	-6	-6		$\frac{wh}{6}$
V	+15	+9	+3	-3	-9	-15		$\frac{wh}{6}$ begin from centre
M	0	15	24	27	24	15	0	$\frac{wh^2}{6}$
α	0	-15	-24	-27	-24	-15	0	$\frac{wh^2}{6EI}$
$\bar{\alpha}$	x	-174	-282	-318	-282	-174	0	$\frac{wh^2}{6EI} \cdot \frac{h}{12}$
ϕ	+615	+441	+159	-159	-441	-615		$\frac{wh^3}{72EI}$ begin from centre
y	0	+615	+1056	1215	+1056	+615	0	$\frac{wh^3}{72EI}\ h$

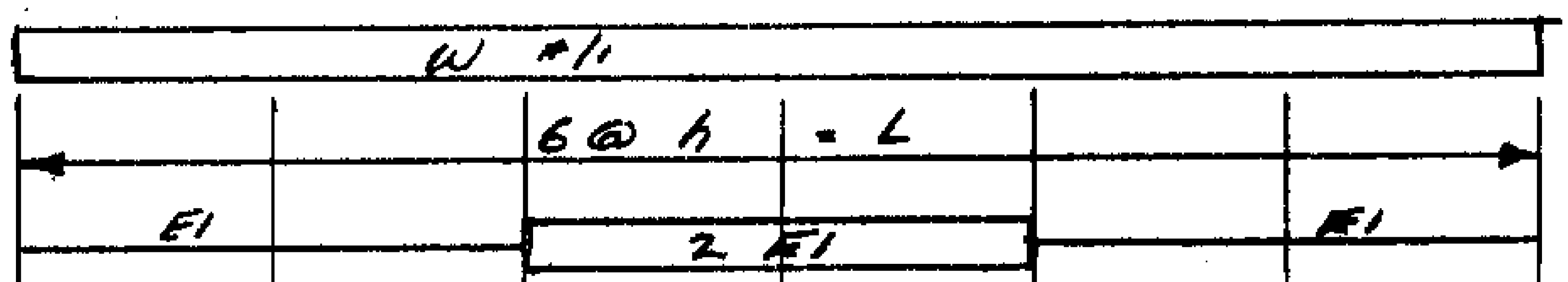
$$y_{℄} = 1215 \times \frac{wh^3 \times h}{72EI} = \frac{5}{384}\frac{wL^4}{EI}$$

Fig. 1-14

Procedure

P - Load at any point - w#/'
$\bar{P}$ - Equivalent concentration - wh - at any interior panel point.
V - Shear in each panel obtained from values of $\bar{P}$.
M - Moment obtained from shear values.
α - ~~Angle change~~ - M/EI — Curvature
$\bar{\alpha}$ - Equivalent angle change concentration
Using parabolic load curve $\frac{h}{12}(a + 10b + c)$
Ø - Slope of panel.
y - Deflection at panel points.

(iii) <u>Simple Beam - Uniform load</u> - EI changed due to stiffening plate along 2 center panels.



Values from P to M are the same as in Fig 1-14

M	0	15	24	27	24	15	0	$wh^2/6$
α		-15	-24 | -12	$-\frac{27}{2}$	-12 | -24	-15		$\frac{wh^2}{6EI}$
τ		-174	-129 | -76.5	-159	-205.5	-174		$\frac{wh^3}{72EI}$
φ	+459	+285	+79.5	-79.5	-285	-459		$\frac{wh^3}{72EI}$ begin from ℄
y	0	459	744	823.5	744	459	0	$\frac{wh^4}{72EI}$

Fig. 1-15

Procedure is same as for previous beam except that two separate angle change concentrations are needed due to discontinuity of M/EI curve.

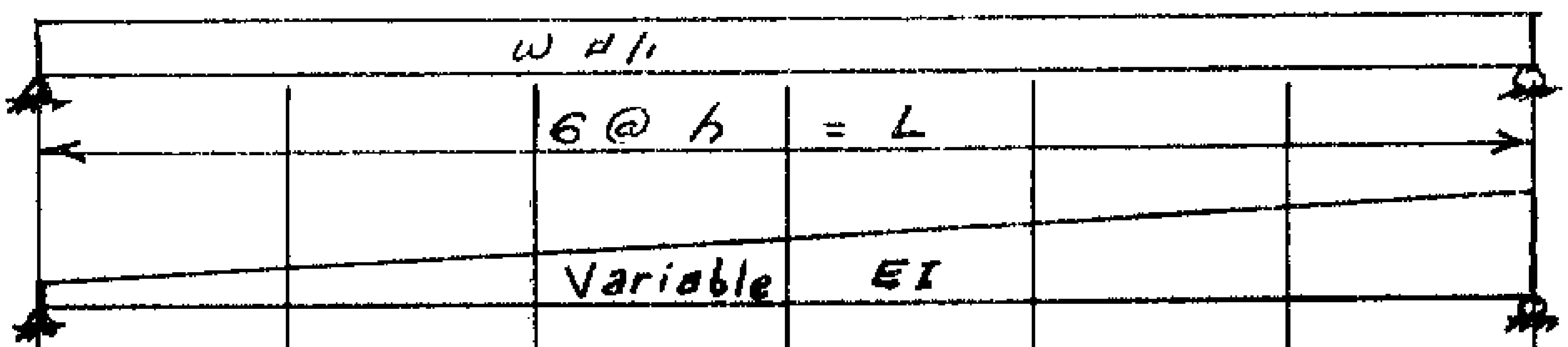
Effect of stiffening plate on M/EI diagram.

Fig 1-16

Resultant M/EI diagram

(iv) <u>Simple beam - Uniform load</u> - variable EI

(see following page)



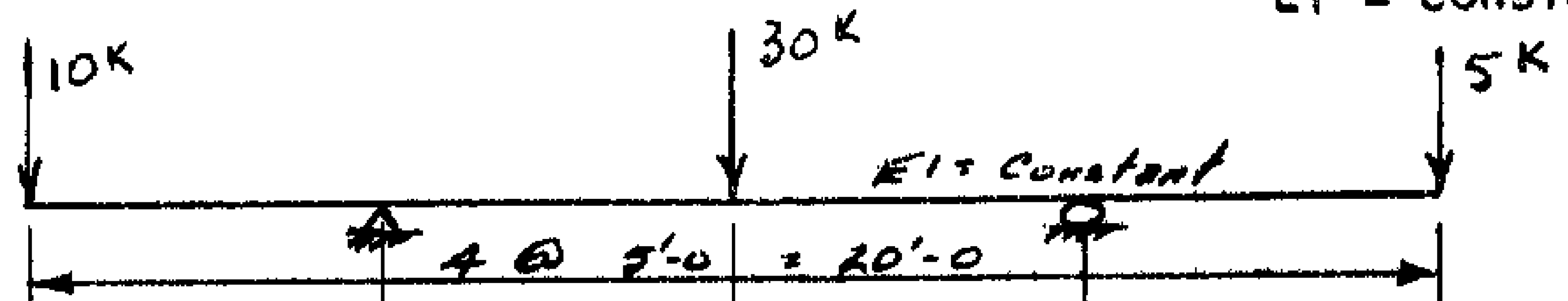
EI	1		2		3		4		5		6		7	EI
P	x		-1		-1		-1		-1		-1		x	w
$\bar{P}$	x		-6		-6		-6		-6		-6		x	$\frac{wh}{6}$
V		+15		+9		+3		-3		-9		-15		$\frac{wh}{6}$ begin from ℄
M	0		+15		+24		+27		+24		+15		0	$\frac{wh^2}{6}$
α	0		$-\frac{15}{2}$		$-\frac{24}{3}$		$-\frac{27}{4}$		$-\frac{24}{5}$		$-\frac{15}{6}$		0	$\frac{wh^2}{6EI}$
α	0		-450		-480		-405		-288		-150		0	$\frac{wh^2}{6EI} \times \frac{1}{60}$
$\bar{\alpha}$	x		-4980		-5625		-4818		-3435		-1788		0	$\frac{wh^2}{6EI} \times \frac{1}{60} \times \frac{h}{12}$
Ø		Assumed +12 000		+7020		+1365		-3453		-6888		-8676		$\frac{wh^3}{72 \times 60\,EI} \times h$
y_1	0		+12000		+19020		+20385		+16932		+10044		+1368	$\frac{wh^3}{72 \times 60\,EI} \times h$
y_c	0		-228		-456		-684		-912		-1140		-1368	$\frac{wh^3}{72 \times 60\,EI} \times h$
y_F	0		+11772		+18564		+19701		+16020		+8904		0	$\frac{wh^4}{72 \times 60\,EI}$

Fig 1-17

Procedure

P - Load at any panel point = w#/' (end reactions not required).
$\bar{P}$ - Equivalent concentration = wh at any panel point.
V - Shear in each panel obtained from $\bar{P}$ values. Load symmetrical about ℄, therefore, begin to compute values of shear from ℄ and proceed each way from ℄ as before.
M - Moment obtained from shear values by the method that the change in BM between two sections on a beam equals the area under SF curve between the two sections.
α - Angle change at any panel point = M/EI at the panel point.
$\bar{\alpha}$ - Equivalent angle change concentration using the equivalent concentration parabolic load curve, i.e. equivalent concentration = h/12 (a + 10b + c).
Ø - Slope of panel (analogous to shear). Assume value of shear in first panel, compute shears in remaining panels and compute deflection. Know boundary conditions. Therefore apply linear correction to deflection.
y - Deflection at panel points (analogous to BM).
  y_1 = deflections at panel points based on assumed value of shear in first panel.
  y_c = linear correction to deflections at panel points.
  y_f = true values of deflections at panel points.

(v) <u>Simple beam</u> - with overhanging ends - series of concentrated loads - EI = constant.



P	-10		×		-30		×		-5	Kips
V_1		-10		Assume +25		-5		+5		K
M_1	0		-10		+15		+10 (-5) Actual Value		0	$\times 5^{1K}$
M_c	0		0		$-7\frac{1}{2}$		-15		0	$\times 5^{1K}$
M_F	0		-10		$+7\frac{1}{2}$		-5		0	$\times 5^{1K}$
V_F		-10		$+17\frac{1}{2}$		$-12\frac{1}{2}$		+5		K
$\bar{\alpha}$	×		$+32\frac{1}{2}$		-15		$+12\frac{1}{2}$		×	$\times 5'^K \times \frac{5}{6EI}$
ϕ		Assume -40		$-7\frac{1}{2}$		$-22\frac{1}{2}$		-10		$\times \frac{25}{6EI}$
y_1	+40		0		$-7\frac{1}{2}$		-30		-40	$\times \frac{125}{6EI}{}'$
y_c	-15		0		+15		+30		+45	$\times \frac{125'}{6EI}$
y_F	+25		0		$+7\frac{1}{2}$		0		+5	$\times \frac{125}{6EI}$

Fig. 1-18

<u>Procedure</u>

P - Load on beam at point of load - reactions not considered.

V_I - Shear force based on assumed value of shear in first interior panel.

M_I - Moment based on V_I.

M_C - Correction moment. (Know moments at the reactions.)

M_F - True value of moments at panel points.

V_F - Correct shear based on correct moments.

$\bar{\alpha}$ - Equivalent angle change concentration by use of trapezoidal load curve $\frac{h}{6}(a + 4b + c)$

Ø - Slope of panel (analogous to shear) based on assumed value of shear in left exterior panel.

y_I - Deflection at panel points based on assumed value of shear in left exterior panel. (Analogous to bending moment.)

y_C - Correction to deflection curve; deflections at reactions are zero.

y_F - Correct value of deflection at panel points.

2. Maxwell's Theorem of Reciprocal Deformation

(2-1)

Take two points A and B on a beam. Maxwell's Theorem of Reciprocal Deformations states that the deflection at point B in the direction D caused by force 1 acting at point A in the direction of C is identical with the deflection which force 1 acting at point B in the direction D would cause at point A in the direction C. The proof of this is as follows.

Let M = BM due to 1 acting at A in direction C
m = BM due to unit load acting at B in direction D
M_1 = BM due to 1 acting at B in direction D
m_1 = BM due to unit load acting at A in direction C.

The deflection at B in direction D is $\Delta B = \int_L \frac{Mm\,dx}{EI}$

The deflection at A in direction C is $\Delta A = \int_L \frac{M_1 m_1\,dx}{EI}$

m is the BM due to a unit load acting at B in the direction C. Therefore the BM, M_1 due to a force 1 acting at B in the direction C is force 1 times m.

Therefore M_1 = (Force 1)m

By a similar argument M = (Force 1)m_1

Then

$$\Delta B = \int_L \frac{Mm\,dx}{EI} = \int_L \frac{m_1 m(\text{Force } 1)\,dx}{EI}$$

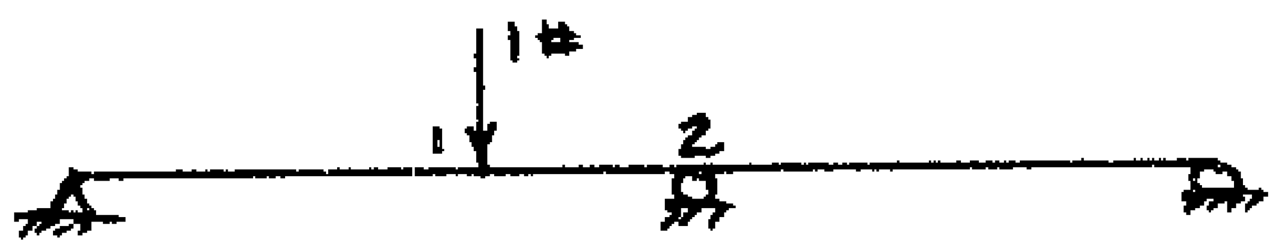
and

$$\Delta A = \int_L \frac{M_1 m_1\,dx}{EI} = \int_L \frac{m_1 m(\text{Force } 1)\,dx}{EI}$$

As can be seen $\Delta A = \Delta B$ which proves Maxwell's Reciprocal Deflection Theorem.

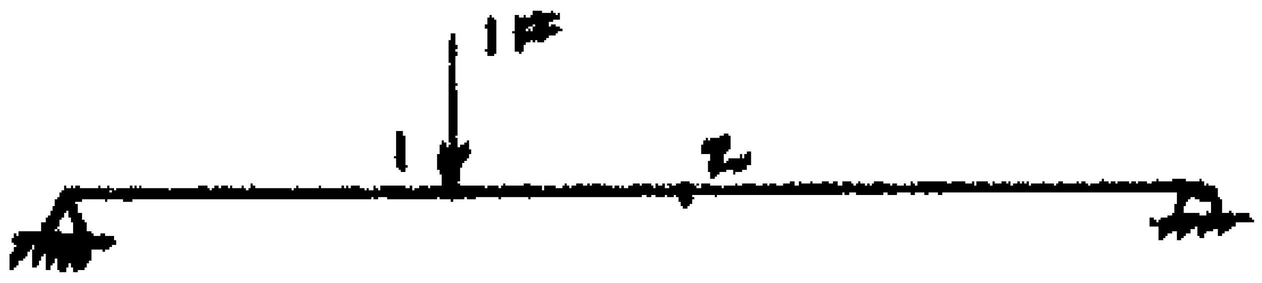
(2-2) Use of Maxwell's Reciprocal Theorem to find Influence Lines for Statically Indeterminate Beams.

Consider a two-span continuous beam.



EI = constant

The influence line for the reaction at 2 is defined as the reaction at 2 due to a load acting at any point on the span length. The following procedure is used in determining the influence line for the reaction at 2 using Maxwell's Reciprocal Theorem.



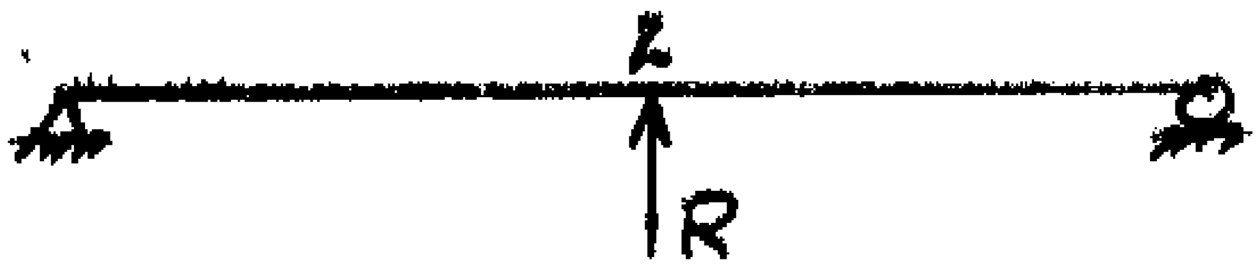
(I) Remove the reaction

Δ_{2-1} = deflection at 2 due to a unit load at 1.

(II) Place the unit load at 2.

Δ_{2-2} = deflection at 2 due to a unit load at 2.

(III) Remove unit load and replace by reaction due to unit load.



$R\Delta_{2-2}$ is the deflection at 2 due to the reaction R.

From Maxwell's Reciprocal Theorem

$$\Delta_{2-1} = R\Delta_{2-2}$$

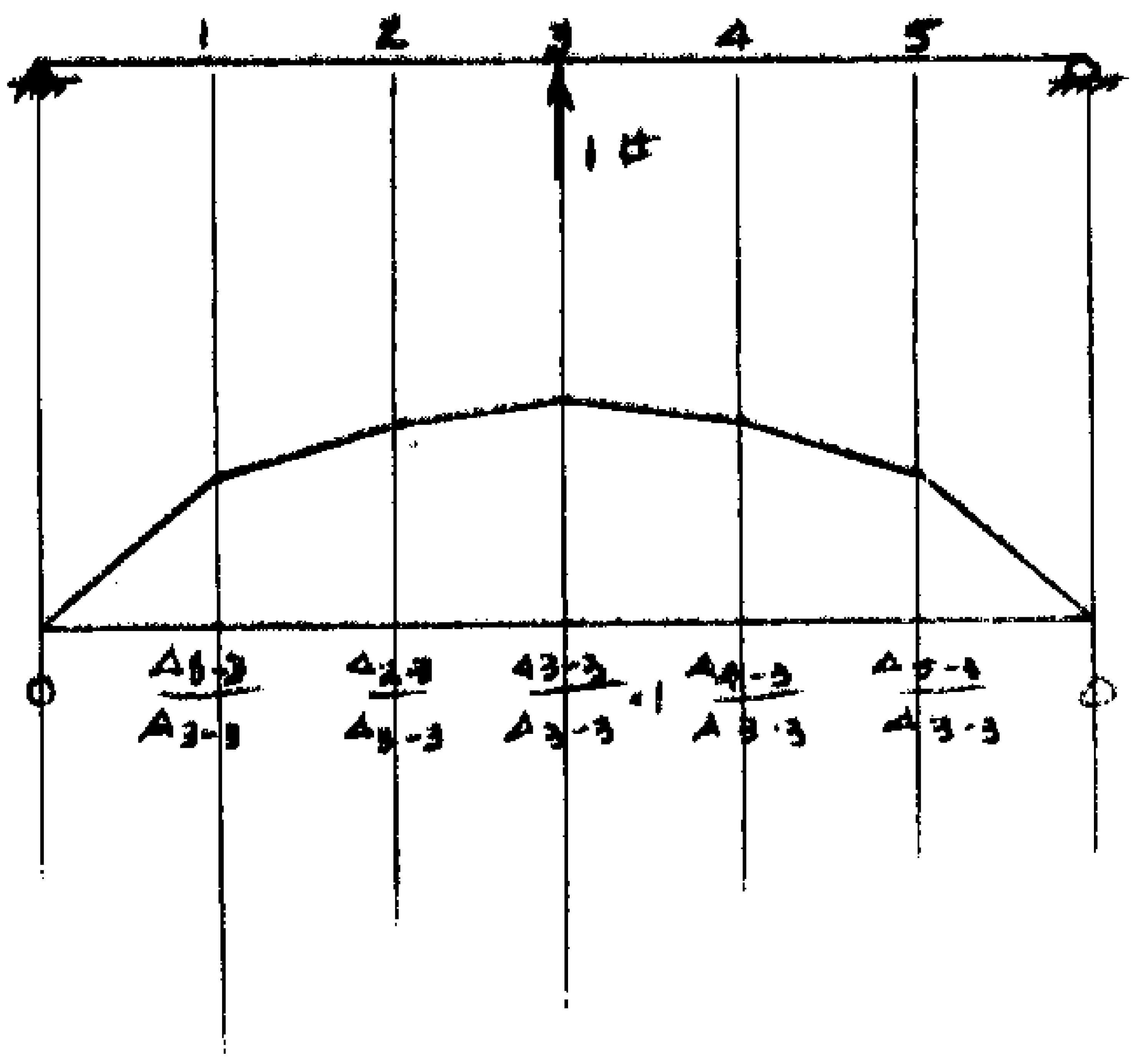
and

$$R = \frac{\Delta_{2-1}}{\Delta_{2-2}}$$ which gives ordinates for the influence line.

Example I

Method:

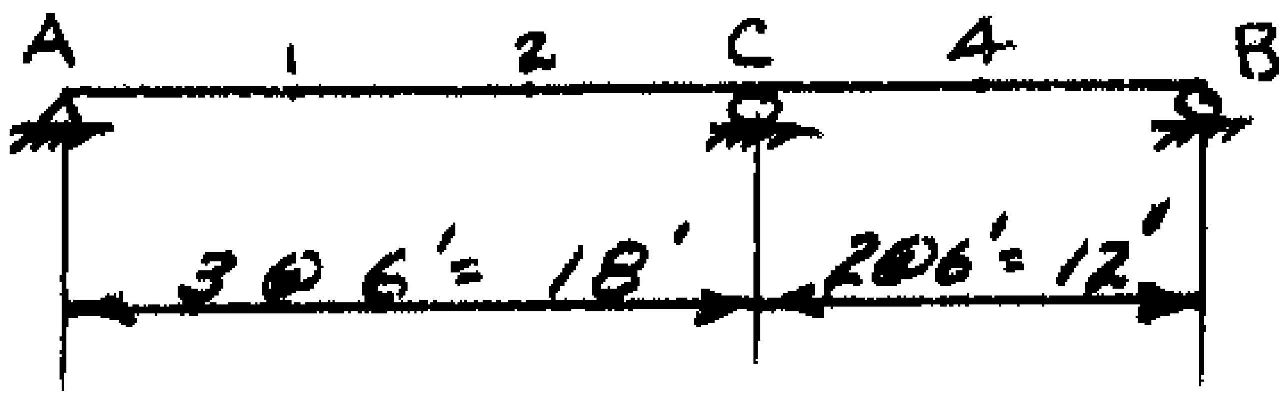
1. Place a unit load at 3.
2. Find deflections at all points due to this unit load at 3.
3. From Maxwell's Reciprocal Theorem $\Delta_{1-3} = \Delta_{3-1}$, that is, a deflection at 1 due to a unit load at 3 is equal to a deflection at 3 due to a unit load at 1.
4. The ordinates for the influence line are obtained by dividing the deflections at various points by the deflection at 3.



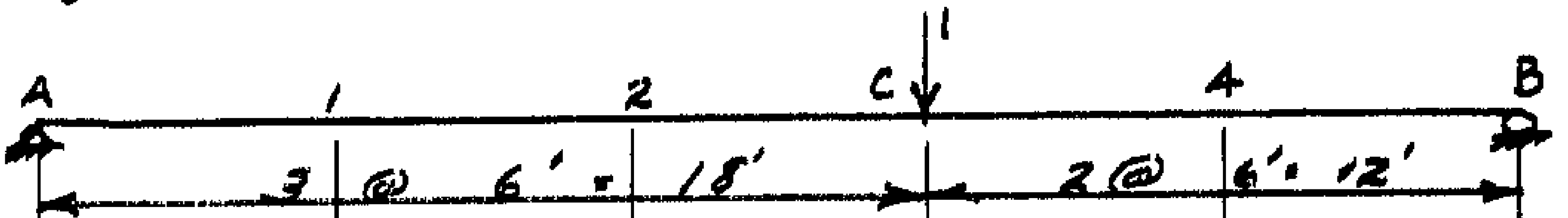
Example 2

Find the influence line for the interior reaction.

EI = constant

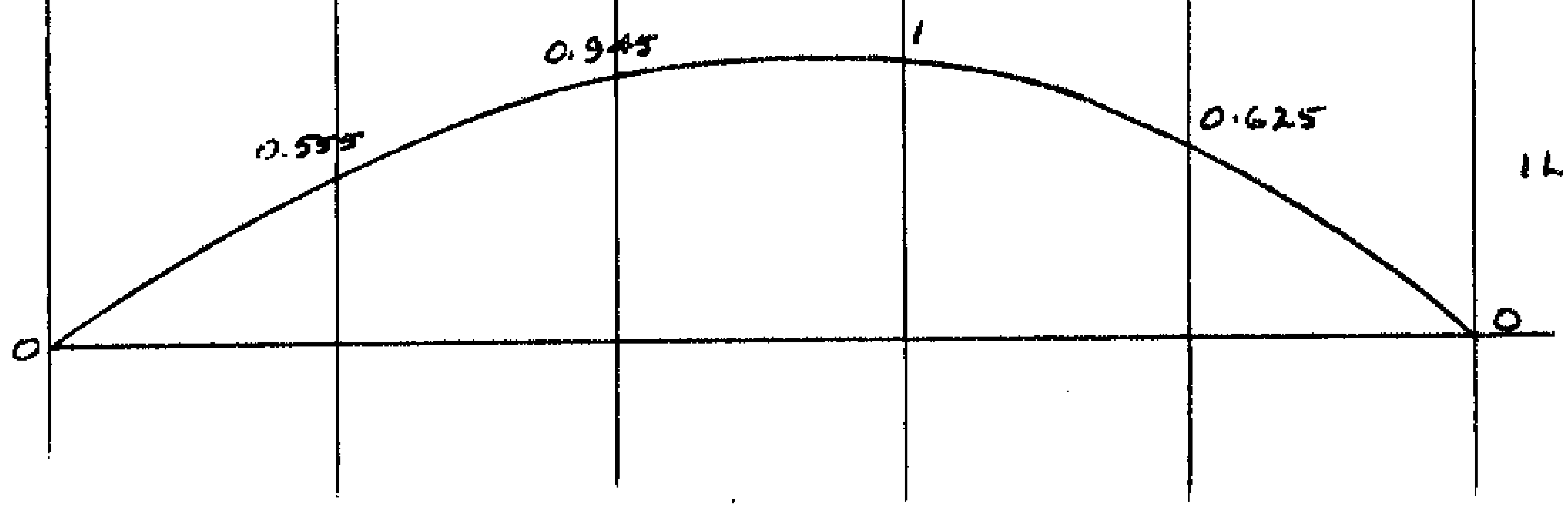


The solution is obtained by removing reaction C and replacing it with a unit load. The deflection at all points is determined then divided by the deflection at C to give the ordinates of the influence line.



EI = Constant

	A	A–1	1	1–2	2	2–C	C	C–4	4	4–B	B	
P	×		0		0		−1		0		×	
V		Assume +2		+2		+2		+1		+1		$\times \frac{1}{3}$ Correction constant
M_1	0		+2		+4		+6		+7		+8	$\times \frac{6}{3}$
M_C	0		−8/5		−16/5		−24/5		−32/5		−8	$\times 6/3$
M_F	0		+2/5		+4/5		+6/5		+3/5		0	$\times 6/3$
1F	0		+2		+4		+6		+3		0	$\times \frac{6}{3\times5}$
α	0		−2		−4		−6		−3		0	$\times \frac{6}{15EI}$
$\bar{\alpha}$	×		−12		−24		−31		−18		×	$\times \frac{6}{15EI} \times \frac{6}{6}$
ϕ		Assume +30		+18		−6		−37		−55		$\times \frac{6}{15EI}$
y_1	0		+30		+48		+42		+5		−50	$\times \frac{6}{15EI} \times 6$
y_C	0		+10		+20		+30		+40		+50	$\times \frac{36}{15EI}$
y_F	0		+40		+68		+72		+45		0	$\times \frac{36}{15EI}$
R_C	0		$\frac{40}{72}$		$\frac{68}{72}$		$\frac{72}{72}$		$\frac{45}{72}$		0	Ordinates
	0		0.555		0.945		1		0.625		0	

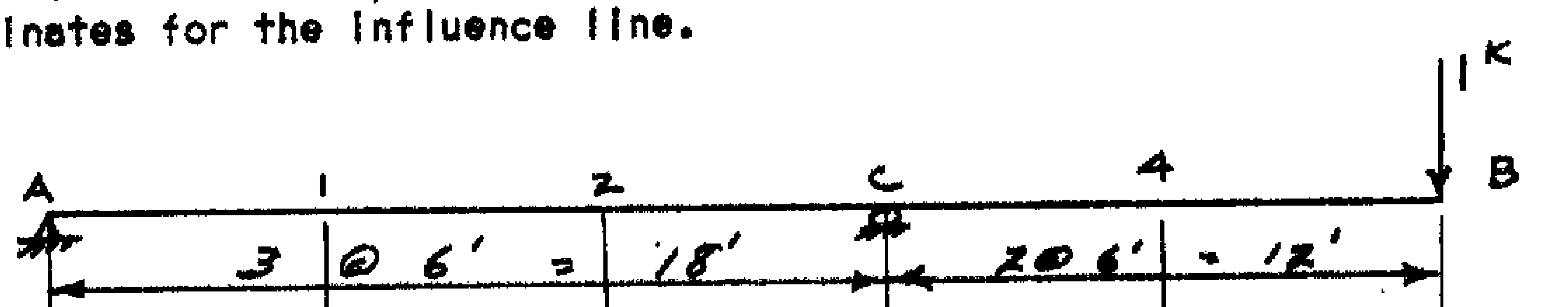


IL for R_c

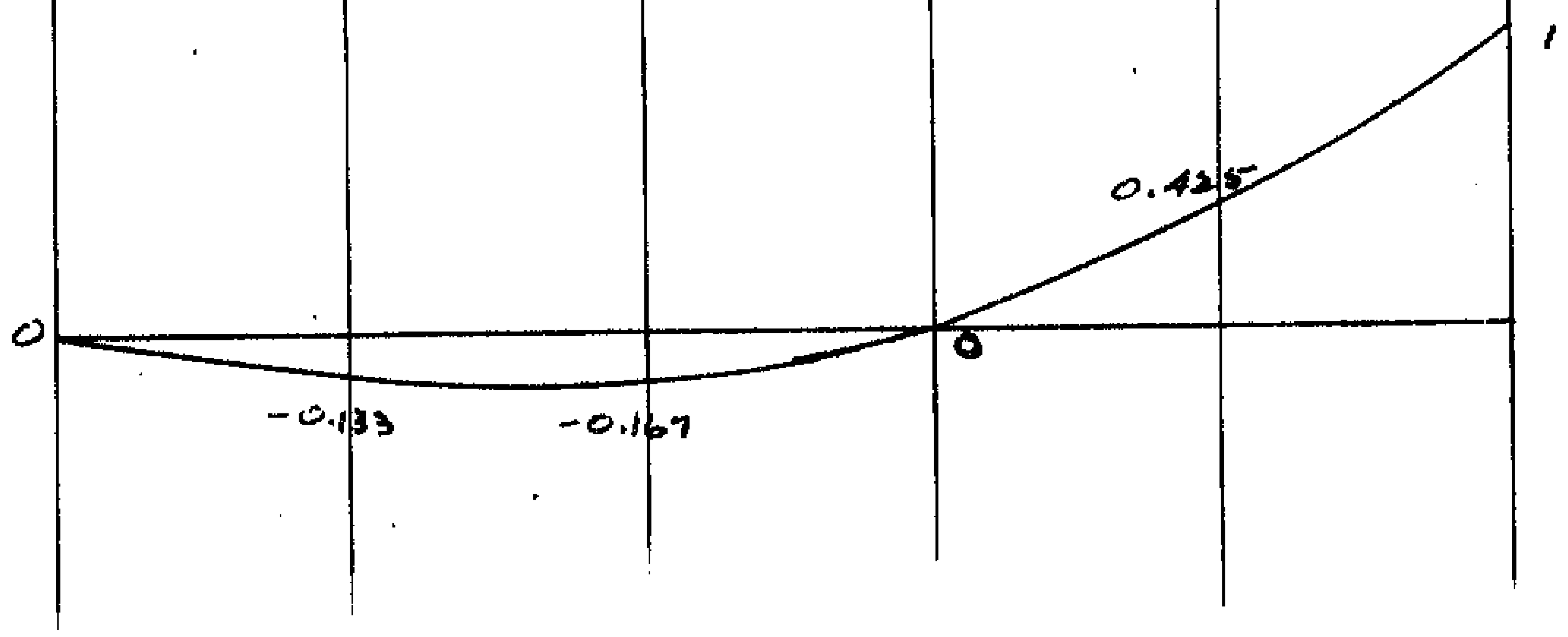
Example 3

Influence line for right-hand reaction of the same beam.

Remove right hand reaction and replace it with a unit load. Find the deflections at all points and divide by the deflection at B to obtain the ordinates for the influence line.



	A		1		2		C		4		B	
P	✗		0		0		✗		0		-1	k
V		Assume +2		+2		+2		+1		+1		k
M_1	0		+2		+4		+6		-1		0	$\times 6^{1k}$
M_c	0		[illegible]		$-\frac{16}{3}$		-8		0		0	$\times 6^{1k}$
M_F	0		$-\frac{2}{3}$		$-\frac{4}{3}$		-2		-1		0	$\times 6^{1k}$
M_F	0		-2		-4		-6		-3		0	$\times \frac{6}{3}$ 1k
α	0		+2		+4		+6		+3		0	$\times \frac{6}{3EI}$ rad.
$\bar{\alpha}$	✗		+12		+24		+31		+18		✗	$\times \frac{6}{3EI} \times \frac{6}{6}$ rad.
ϕ		Assume -30		-18		+6		+37		+55		$\times \frac{6}{3EI}$ rad/1
y_1	0		-30		-48		-42		-5		+50	$\times \frac{6}{3EI} \times 6$ ft.
y_c	0		+14		+28		+42		+56		+70	$\times \frac{6}{3EI} \times 6$ ft.
y_F	0		-16		-20		0		+51		+120	$\times \frac{6}{3EI} \times 6$ ft.
R_B	0		$\frac{-16}{120}$		$\frac{-20}{120}$		0		$\frac{+51}{120}$		$\frac{+120}{120}$	Ordinates
	0		-0.133		-0.167		0		+0.425		1	

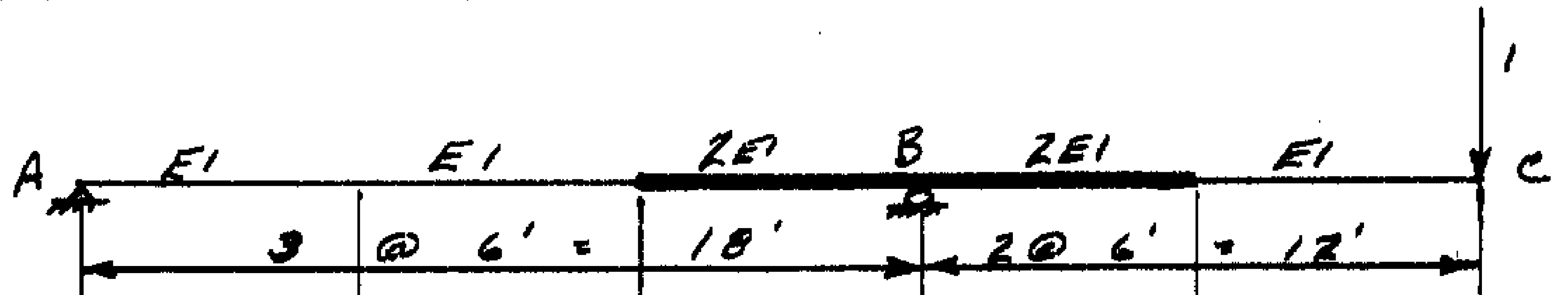


IL for R_B

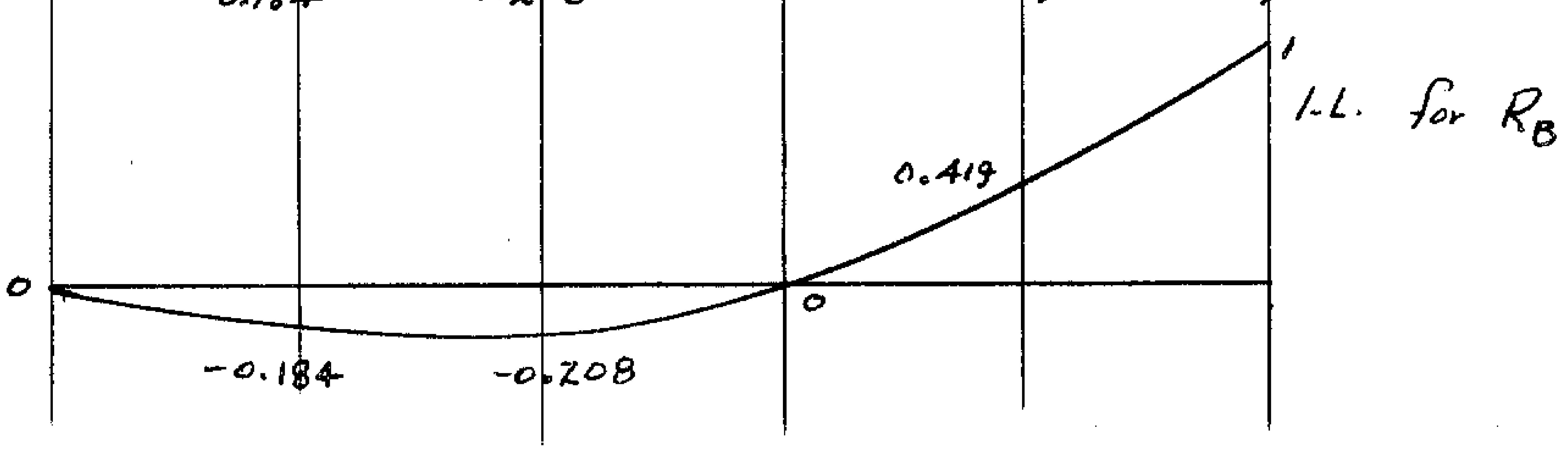
Example 4

Influence line for right-hand reaction with cover plates.

The reaction is removed and replaced with a unit load. The procedure is the same as before.



P	1	0	0	x	0	-1	
V	Assume +2	+2	+2	+1	+1		
M_1	0	+2	+4	+6	-1	0	$\times 6$
M_a	0	$-\frac{8}{3}$	$-\frac{16}{3}$	-8	0	0	$\times 6$
M_b	0	$-\frac{2}{3}$	$-\frac{4}{3}$	-2	-1	0	$\times 6$
M/EI	0	-2	-4	-6	-3	0	$\times \frac{6}{3}$
α	0	2	4	6	3	0	$\times \frac{6}{3EI}$
Z	x	12	$\frac{10+7}{17}$	$\frac{31}{2}$	12	x	$\times \frac{6}{3EI} \times \frac{6}{6}$
$\bar{Z}$	x	24	34	31	24	x	$\times \frac{6}{3EI} \times \frac{6}{6} \times \frac{1}{2}$
ϕ	Assume -30	-6	+28	+59	+83		$\times \frac{6}{3EI} \times \frac{6}{6} \times \frac{1}{2}$
y_1	0	-30	-36	-8	+51	+134	$\times \frac{6}{3EI} \times \frac{6}{6} \times \frac{1}{2} \times 6$
y_c	0	$+\frac{8}{3}$	$+\frac{16}{3}$	+8	$+\frac{32}{3}$	$+\frac{40}{3}$	$\times \frac{6}{3EI} \times \frac{6}{6} \times \frac{1}{2} \times 6$
y	0	$-\frac{82}{3}$	$-\frac{92}{3}$	0	$\frac{185}{3}$	$\frac{442}{3}$	$\times \frac{6}{3EI} \times \frac{6}{6} \times \frac{1}{2} \times 6$
y	0	-82	-92	0	185	442	$\times \frac{6}{3EI} \times \frac{6}{6} \times \frac{1}{2} \times \frac{6}{3}$
R_B	0	$\frac{-82}{442}$	$\frac{-92}{442}$	0	$\frac{185}{442}$	$\frac{442}{442}$	Ordinates
	0	-0.184	-0.208	0	0.419	1	



(2-3) Muller-Breslau's Principle

This method can be used when the structure is indeterminate to more than one degree. The method is as follows:

1. Give the point a unit displacement and work out the moments due to differential movements holding the joints solid.

2. Release the joints and balance by moment distribution.

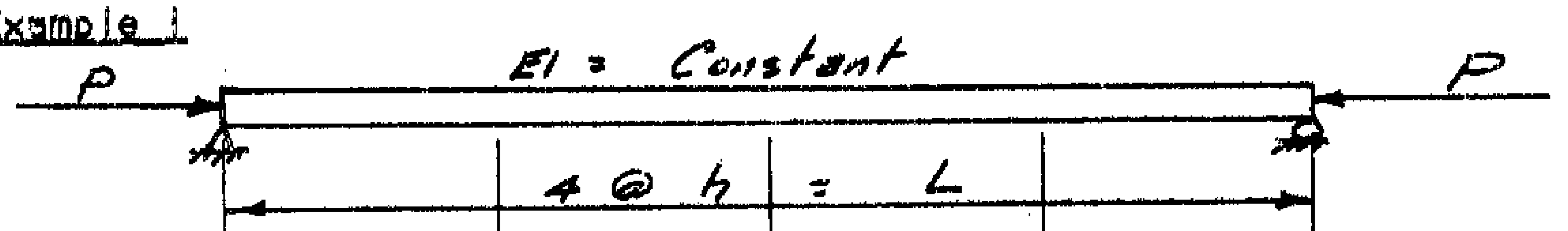
3. α can be worked out knowing the stiffnesses.

3. Critical Loads on Columns

(3-1) The buckling of columns as well as the buckling of beams can be treated by numerical procedures. Buckling of columns can be defined in several ways. Here the buckling of a column will be defined in terms of the critical load as that compression load which when applied to a compression member, the member will just support that load and remain in stable equilibrium. On the addition of a small increment of load, however, the system will fail.

If a column is put in a deflected form by means of a compressive force, the deflected form will be a simple configuration. Therefore, if we assume a configuration and check to see if the configuration is the same for the deflected form, and if the two configurations are the same, we have the correct answer. The following example will illustrate the use of the procedure.

Example 1



Δ	0	+1	+1.5	+1	0	$\times a$
M	0	+1	+1.5	+1	0	$\times Pa$
α	0	−1	−1.5	−1	0	$\times \frac{Pa}{EI}$
$\bar{\alpha}$		−11.5	−17	−11.5		$\times \frac{Pa}{EI} \times \frac{h}{12}$
ϕ	Assume +20	+8.5	−8.5	−20		$\times \frac{Pa}{EI} \times \frac{h}{12}$
y	0	+20	+28.5	+20	0	$\times \frac{Pa}{EI} \times \frac{h}{12} \times h$

Procedure

1. Assume configuration by assuming values of "Δ".
2. Bending moment equals $P \times \Delta$.
3. Values of α are found by dividing the bending moment at each section by EI.
4. Values of $\bar{\alpha}$ are found by the use of equivalent concentrations and Simpson's Rule as before.
5. The values of $\emptyset$ are found by assuming a value for the slope in the first panel and on this basis the remainder are calculated as before.

6. The values of the deflection "y" based on the assumed slope in the first panel are determined.

We now have both the assumed and the resultant configurations. It is now necessary to check one against the other. Suppose we check the conditions at the first interior panel point. Then we have -

$$1 \times a = \frac{20 \times P \times a \times h^2}{EI \times 12} \quad \text{and} \quad h = \frac{L}{4}$$

therefore

$$P = \frac{12 \times EI \times 16}{L^2 \times 20} = \frac{9.6\ EI}{L^2}$$

Checking conditions at the centerline, we have

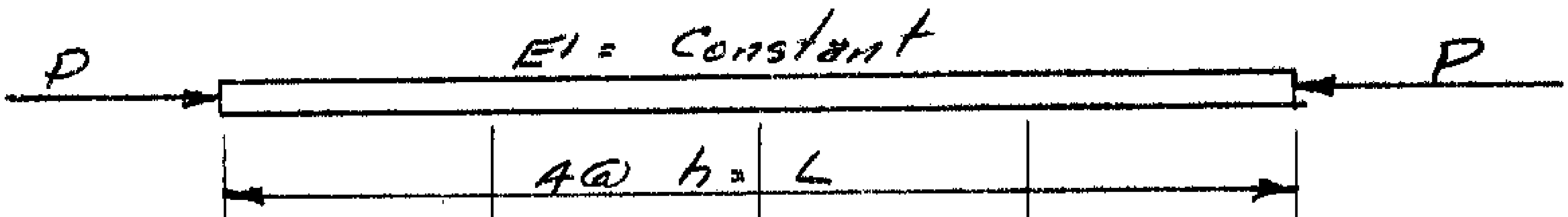
$$P = \frac{1.5 \times 12\ EI \times 16}{28.5 \times L^2} = \frac{10.1\ EI}{L^2}$$

where the assumed value of Δ is $1.5 \times a$.

It is now necessary to repeat the above procedure for a closer approximation of "Δ" . The correct value of $\Delta_{\mathbb{C}}$ may be found by equating:

$$P = \frac{\Delta_{\mathbb{C}} \times 12\ EI \times 16}{28.5\ L^2} = \frac{9.6\ EI}{L^2} \quad \text{or correct } \Delta_{\mathbb{C}} = \frac{28.5}{20} = 1.43$$

From the above we have a closer value of "$\Delta_{\mathbb{C}}$" = 1.43 x a . Using this value in the next trial, we will get better results.



Δ	0		1.0		1.43		1.0		0	$\times a$
M	0		1.0		1.43		1.0		0	$\times Pa$
α	0		-1.0		-1.43		-1.0		0	$\times \frac{Pa}{EI}$
Z	*		-11.43		-16.3		-11.43		*	$\times \frac{Pa}{EI} \times \frac{h}{12}$
ϕ		+19.58		+8.15		-8.15		-19.58		$\times \frac{Pa}{EI} \times \frac{h}{12}$
y	0		19.58		27.73		19.58		0	$\times \frac{Pa}{EI} \times \frac{h^2}{12}$

From above for the deflection at first interior panel point -

$$P = \frac{12 \times EI \times 16}{19.58 \times L^2} = \frac{9.83\ EI}{L^2}$$

Using Δ_t -

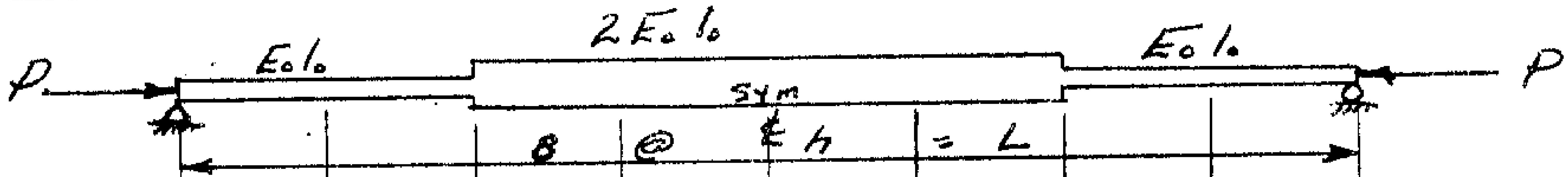
$$P = \frac{12\ EI \times 1.43 \times 16}{27.73 \times L^2} = \frac{9.88\ EI}{L^2}$$

The above procedures hold for critical load only. If $P \neq P_{cr}$, the member does not hold deflected shape and the above procedures do not hold.

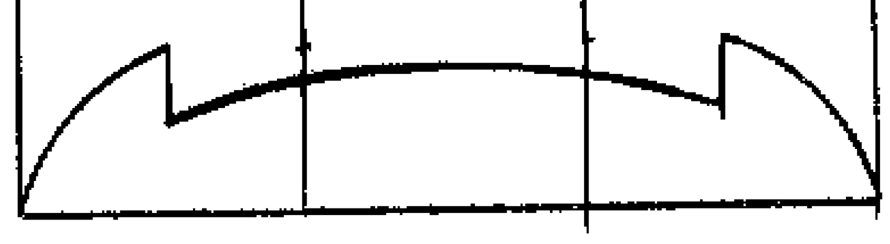
A reasonable approximation for the value of P_{cr} from the second or third trial provided an intelligent guess for the values of Δ is made. If the convergence is slow, several more trials may be necessary. Usually three trials are sufficient for simple cases. Solutions for more complicated cases may be obtained by use of Dinnick's Tables.

Another example will illustrate the use of numerical procedures in solution for P_{cr} for columns with varying cross-section.

Example 2



Δ,	0	+1.0	+1.2	+1.3	+1.35	+1.3	+1.2	+1.0	0	$\times a$
M,	0	+1.0	+1.2	+1.3	+1.35	+1.3	+1.2	+1.0	0	$\times Pa$
α,	0	-1.0	-1.2 -0.4	-0.65	-0.675	-0.65	-0.4 -1.2	-1.0	0	$\times \frac{Pa}{E_o I_o}$
Z,	⌧	-11.2	-7.2 -3.91	-7.78	-8.05	-7.78	-3.91 -7.2	-11.2	⌧	$\times \frac{Pa}{E_o I_o} \times \frac{h}{12}$
ϕ,	+33.92	+22.72	11.81	+4.03	-4.03	-11.81	-22.72	-33.92		$\times \frac{Pa}{E_o I_o} \times \frac{h}{12}$
y,	0	+33.92	+56.62	+68.45	+72.48	+68.45	+56.62	+33.92	0	$\times \frac{Pa}{E_o I_o} \times \frac{h^2}{12}$



$\frac{M}{E.I.}$ Diag.

Checking at first interior panel point - $33.92 \times \frac{P \times a \times h^2}{12\ E_o\ I_o} = 1 \times a$

and $P_{cr} = \frac{12\ E_o\ I_o \times 64}{33.92\ \ L^2} = \frac{22.8\ E_o\ I_o}{L^2}$

This is very much in error. Repeat for new values of Δ and check as before.

*see page 25

Checking, $\quad 1.0 \times a = \dfrac{44.38 \times P \times a \times L^2}{12 \times E_o I_o \times 64}$ and $P_{cr} = \dfrac{17.4\ E_o I_o}{L^2}$

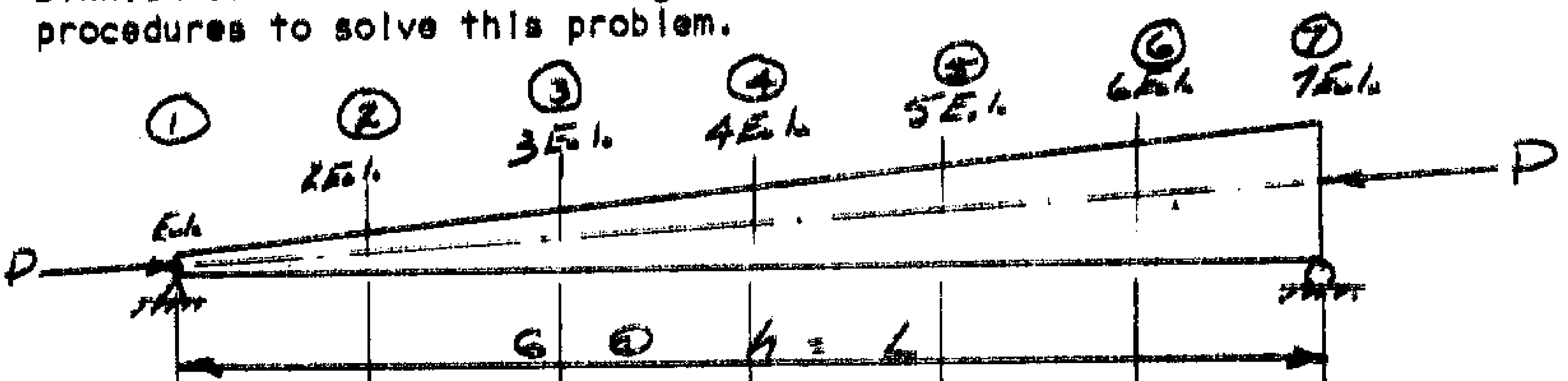
This value of P_{cr} is within 6% of actual value.

Checking at center, $P_{cr} = \dfrac{16.2\ E_o I_o}{L^2}$.

From this example it is seen that when a cover plate is placed over center 50% column length, it is almost as effective as having cover plates over full length. In fact this type of column is about 85% as efficient as if the cover plates are extended over the full length.

(3-2) Columns with Uniformly Varying Moment of Inertia.

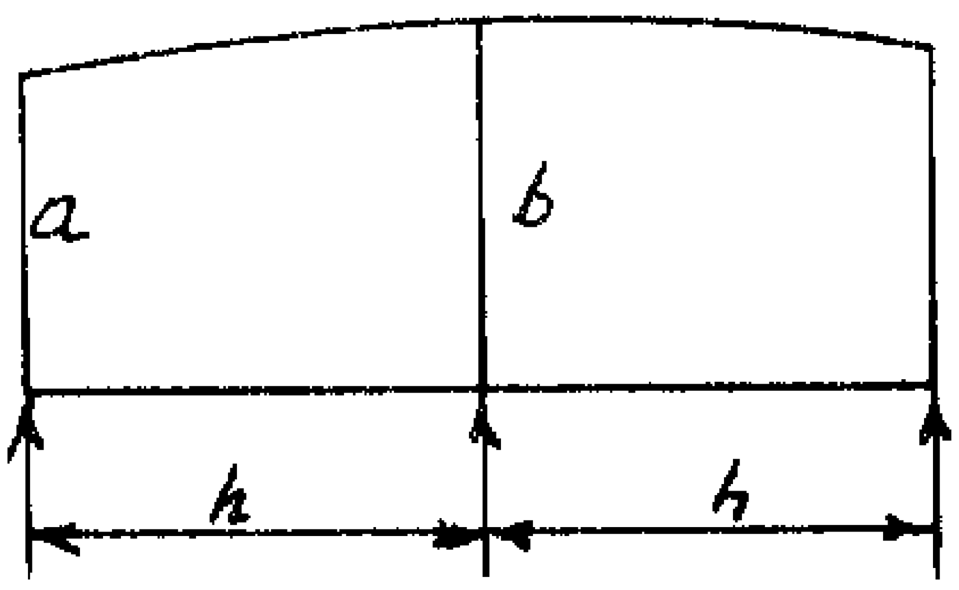
In columns with uniformly varying moment of inertia the numerical procedures show their advantages since columns of this type cannot be solved by the use of Dinnick Tables. The following example will demonstrate the use of the numerical procedures to solve this problem.



	①	②	③	④	⑤	⑥	⑦	
Δ_1	0	+1	+2	+2	+1.4	+0.8	0	$\times a$
M_1	0	+1	+2	+2	+1.4	+0.8	0	$\times Pa$
α	0	-0.5	-0.67	-0.5	-0.28	-0.133	0	$\times \frac{Pa}{E_oI_o}$
$\bar{\alpha}$	x	-5.67	-7.70	-5.95	-3.43	-1.61	x	$\times \frac{Pa}{E_oI_o} \cdot \frac{h}{12}$
ϕ	Assume +16	+10.33	+2.63	-3.32	-6.25	-8.36		$\times \frac{Pah}{12E_oI_o}$
y_1	0	+16	+26.33	+28.96	+25.64	+18.89	+10.53	$\times \frac{Pah^2}{12E_oI_o}$
y_a	0	-1.76	-3.52	-5.28	-7.04	-8.80	-10.53	$\times \frac{Pah^2}{12E_oI_o}$
y	0	+14.24	+22.81	+23.68	+18.60	+10.09	0	$\times \frac{Pah^2}{12E_oI_o}$
Δ_2	0	+1	+1.60	+1.66	+1.30	+0.71	0	$\times a$

The problem is to find the load that is critical in buckling for the beam shown which has a moment of inertia varying uniformly from I_o at one end to $7\ I_o$ at the other. Much work can often be saved if the buckled shape of the column is visualized and an intelligent guess is made as to the deflections Δ_1. These deflections are then guessed and all multiplied by an arbitrary factor "a" which could be feet, inches, etc. The line M is then calculated by $M = P\ \Delta_1$. The line α is found by dividing the M at each panel point by the EI at that point. E.G. at point (3) $\alpha = \dfrac{2Pa}{3\ E_oI_o} = 0.67 \times Pa/E_oI_o$

The values of $\bar{\alpha}$ are found by the use of equivalent concentrations and Simpson's Rule. i.e.



$\frac{h}{24}(7a+6b-c)$ $\quad$ $\frac{h}{12}(a+10b+c)$ $\quad$ $\frac{h}{24}(7c+6b-a)$

The values for α at the various panel points are the values used for a, b and c. It is necessary to assume value for the slope so a value of +16 is used between panel points (1) and (2). Then on this basis the other slopes are calculated. For example, the slope between panel points (2) and (3) is $16 + \bar{\alpha}$ at (2) = $16 - 5.67 = 10.33\ Pa.h/12\ E_oI_o$. The deflections y_1, based on the assumed values of $\emptyset$ calculated by $\Sigma\emptyset$ up to the point concerned times h. It is known that the deflection at point (1) = 0 so this is the starting point. E.G. to find y_1 at point (4)

$$\Sigma\ \emptyset \text{ up to point (4)} = 16.00 + 10.33 + 2.63 = 28.96$$

Therefore y_1 at (4) = $28.96\ Pa.h^2/12\ E_oI_o$.

However, in this way y_1 at point (7) turns out to be $10.53\ Pa.h^2/12\ E_oI_o$. Since the support at (7) is non-yielding the deflection there must be 0. This error is due to the fact that the values of $\emptyset$ were merely assumed ones and necessitates the application of a linear correction which will be called y_c. Since the deflection at (7) = 0, $y_c = -y_1 = -10.53$. The values of y_c at the other points are found by simple ratio. The final deflection y is found by $y = y_1 + y_c$.

If these final deflections (y) are equal to the ones originally assumed (Δ_1), then the original assumptions were correct and $\Delta_1\ a = y(Pa.h^2/12\ E_oI_o)$ at every point on the column. Checking this at point (2)

$$1 \times a = \frac{12.24\ Pa.h^2}{12\ E_oI_o} \qquad \text{and} \qquad h = \frac{L}{6}$$

$$P = \frac{12\ E_oI_o \times 36}{14.24\ \ L^2} = \frac{30.3\ E_oI_o}{L^2}$$

At point (3)

$$2a = \frac{22.81\ Pa.h^2}{12\ E_o I_o}$$

$$P = \frac{2 \times 12 \times 36\ E_o I_o}{22.81\ \ L^2} = \frac{37.8\ E_o I_o}{L^2}$$

In order to get a more accurate value for P the procedure is repeated using new values for Δ which will be called Δ_2. These values are found by ratio as follows:

At point (1) let $\Delta_2 = +1$

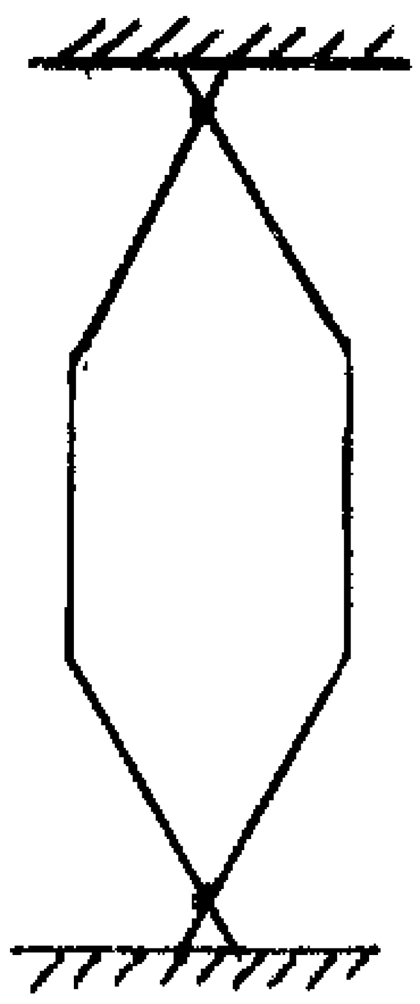
At point (2) $\quad \Delta_2 = \frac{y \text{ at } (2)}{y \text{ at } (1)} = \frac{22.81}{14.24} = +1.60$

At point (3) $\quad \Delta_2 = \frac{y \text{ at } (3)}{y \text{ at } (1)} = \frac{23.68}{14.24} = +1.66$

and so forth. The remainder of this problem will be left as an exercise for the reader. The value of P obtained by this second trial will be considerably more accurate than the value obtained by the first trial and P may be found to any degree of accuracy depending on how many trials are made.

Let us assume that $P = \frac{30.3\ E_o I_o}{L^2}$, although this is being conservative. A column with a moment of inertia of I_o would have a critical load of $P_{cr} = \pi^2 E_o I_o / L^2 = 9.87\ E_o I_o / L^2$. It is interesting to note that a column with moment of inertia varying from I_o to $7 I_o$ has a critical load more than three times as great as the column with moment of inertia I_o. Thus, in calculating the critical load it would be more correct to use the average I than to use the minimum I. It can easily be seen from this problem and the previous one where a cover plate was put over the centre section of the column that the moment of inertia at the centre of the column is the important thing.

One problem which arises when a column of varying EI is used is the determination of the allowable stress on the column. Formulae for allowable stresses on columns usually give the allowable stress as a function of the L/r ratio. However, the expression L/r begins to lose its meaning when a column of the shape shown is encountered.



One possible solution to this problem is to find the critical load of the column shown and then to find the column with the same length but of uniform EI which will have the same critical load as the column shown.

The L/r ratio of this column with uniform EI is then used as the L/r ratio of the column shown for the purpose of calculating the allowable stress.

This discussion is not meant to be a course in numerical procedures but is meant to outline the basic principles used and to indicate some of the problems to which these procedures may be applied. Since many of these simpler problems may be solved more easily by other means, the numerical procedures do not show their true advantage. However, they are extremely useful in many more complicated problems, e.g. design of buildings to withstand earthquake vibrations.

*Diagram for top of page 22

									Sym ℄	
Δ_2	0		1.0		1.67		2.02		2.15	$\times a$
M_2	0		1.0		1.67		2.02		2.15	$\times Pa$
α_2	0		-1.0		-1.67		-2.02		-2.15	$\times \frac{Pa}{E_0I_0}$
$\mathcal{I}_2$	×		-11.67		-8.85 / -5.43		-12.02		-12.82	$\times \frac{Pah}{12E_0I_0}$
ϕ_2		+44.38		+32.71		+18.43		+6.41		$\times \frac{Pah}{12E_0I_0}$
y_2	0		+44.38		+77.09		+95.52		+101.95	$\times \frac{Pah^2}{12E_0I_0}$