Recognizing that the velocity at A is $\dot{x} = a\dot{\theta}$, the above equation becomes

$$T = \frac{1}{2}\left(\frac{J + m_v b^2 + \frac{1}{3}m_s b^2}{a^2}\right)\dot{x}^2$$

Thus the effective mass at A is

$$m_A = \left(\frac{J + m_v b^2 + \frac{1}{3}m_s b^2}{a^2}\right)$$

If the push rod is now reduced to a spring and an additional mass at the end A, the entire system is reduced to a single spring and a mass as shown in Fig. 2.2-5.

EXAMPLE 2.2-6

A simply supported beam of total mass m has a concentrated mass M at midspan. Determine the effective mass of the system at midspan and find its fundamental frequency. The deflection under the load due to a concentrated force P applied at midspan is $Pl^3/48EI$. (See Fig. 2.2-6 and table of stiffness at end of chapter.)

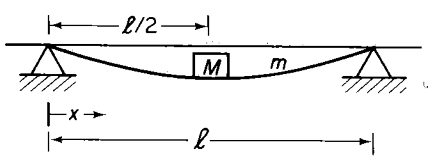

Figure 2.2-6.

Solution: We will assume the deflection of the beam to be that due to a concentrated load at midspan or

$$y = y_{max}\left[\frac{3x}{l} - 4\left(\frac{x}{l}\right)^3\right] \qquad \left(\frac{x}{l} \leqslant \frac{1}{2}\right)$$

The maximum kinetic energy of the beam itself is then

$$T_{max} = \frac{1}{2}\int_0^{l/2}\frac{2m}{l}\left\{\dot{y}_{max}\left[\frac{3x}{l} - 4\left(\frac{x}{l}\right)^3\right]\right\}^2 dx = \frac{1}{2}(0.4857\ \text{m})\dot{y}^2_{max}$$

The effective mass at midspan is then equal to

$$m_{eff} = M + 0.4857\ \text{m}$$

and its natural frequency becomes

$$\omega_n = \sqrt{\frac{48EI}{l^3(M + 0.4857\ \text{m})}}$$

2.3 VISCOUSLY DAMPED FREE VIBRATION

When a linear system of one degree of freedom is excited, its response will depend on the type of excitation and the damping which is present. The equation of motion will in general be of the form

$$m\ddot{x} + F_d + kx = F(t) \tag{2.3-1}$$

where $F(t)$ is the excitation and F_d the damping force. Although the actual description of the damping force is difficult, ideal damping models can be assumed that will often result in satisfactory prediction of the response. Of these models, the viscous damping force, proportional to the velocity, leads to the simplest mathematical treatment.

Viscous damping force is expressed by the equation

$$F_d = c\dot{x} \tag{2.3-2}$$

where c is a constant of proportionality. Symbolically it is designated by a dashpot as shown in Fig. 2.3-1. From the free-body diagram the equation of motion is seen to be

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{2.3-3}$$

The solution of the above equation has two parts. If $F(t) = 0$, we have the homogeneous differential equation whose solution corresponds physically to that of *free-damped vibration*. With $F(t) \neq 0$, we obtain the particular solution that is due to the excitation irrespective of the homogeneous solution. We will first examine the homogeneous equation that will give us some understanding of the role of damping.

With the homogeneous equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.3-4}$$

the traditional approach is to assume a solution of the form

$$x = e^{st} \tag{2.3-5}$$

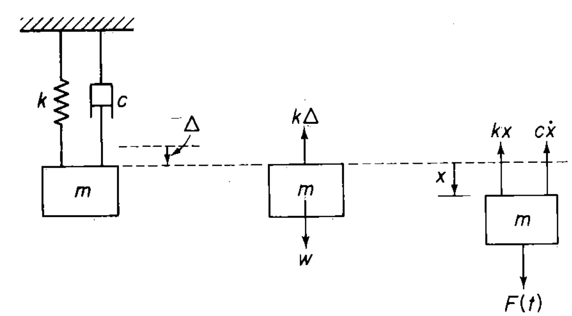

Figure 2.3-1.

where s is a constant. Upon substitution into the differential equation, we obtain

$$(ms^2 + cs + k)e^{st} = 0$$

which is satisfied for all values of t when

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0 \qquad (2.3\text{-}6)$$

Equation (2.3-6), which is known as the *characteristic equation*, has two roots

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad (2.3\text{-}7)$$

Hence, the general solution is given by the equation

$$x = Ae^{s_1 t} + Be^{s_2 t} \qquad (2.3\text{-}8)$$

where A and B are constants to be evaluated from the initial conditions $x(0)$ and $\dot{x}(0)$.

Equation (2.3-7) substituted into (2.3-8) gives

$$x = e^{-(c/2m)t}\left(Ae^{\sqrt{(c/2m)^2 - k/m}\,t} + Be^{-\sqrt{(c/2m)^2 - k/m}\,t}\right) \qquad (2.3\text{-}9)$$

The first term $e^{-(c/2m)t}$ is simply an exponentially decaying function of time. The behavior of the terms in the parentheses, however, depends on whether the numerical value within the radical is positive, zero, or negative.

When the damping term $(c/2m)^2$ is larger than k/m, the exponents in the above equation are real numbers and no oscillations are possible. We refer to this case as *overdamped.*

When the damping term $(c/2m)^2$ is less than k/m, the exponent becomes an imaginary number, $\pm i\sqrt{k/m - (c/2m)^2}\ t$. Since

$$e^{\pm i\sqrt{k/m - (c/2m)^2}\,t} = \cos\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\ t \pm i \sin\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\ t$$

the terms of Eq. (2.3-9) within the parentheses are oscillatory. We refer to this case as *underdamped.*

As a limiting case between the oscillatory and nonoscillatory motion, we define *critical damping* as the value of c which reduces the radical to zero.

It is now advisable to examine these three cases in detail, and in terms of quantities used in practice. We begin with the critical damping.

Critical Damping. For critical damping c_c, the radical in Eq. (2.3-9) is zero.

$$\left(\frac{c_c}{2m}\right)^2 = \frac{k}{m} = \omega_n^2$$

or

$$c_c = 2\sqrt{km} = 2m\omega_n \qquad (2.3\text{-}10)$$

It is convenient to express the value of any damping in terms of the critical damping by the nondimensional ratio

$$\zeta = \frac{c}{c_c} \qquad (2.3\text{-}11)$$

which is called the *damping ratio*. We now express the roots of Eq. (2.3-7) in terms of ζ by noting that

$$\frac{c}{2m} = \zeta\frac{c_c}{2m} = \zeta\omega_n$$

Equation (2.3-7) then becomes

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \qquad (2.3\text{-}12)$$

and the three cases of damping previously discussed now depend on whether ζ is greater than, less than, or equal to unity.

Figure 2.3-2 shows Eq. (2.3-12) plotted in a complex plane with ζ along the horizontal axis. If $\zeta = 0$, Eq. (2.3-12) reduces to $s_{1,2}/\omega_n = \pm i$ so that the roots on the imaginary axis correspond to the undamped case. For $0 \le \zeta \le 1$, Eq. (2.3-12) can be rewritten as

$$\frac{s_{1,2}}{\omega_n} = -\zeta \pm i\sqrt{1 - \zeta^2}\ .$$

The roots s_1 and s_2 are then conjugate complex points on a circular arc

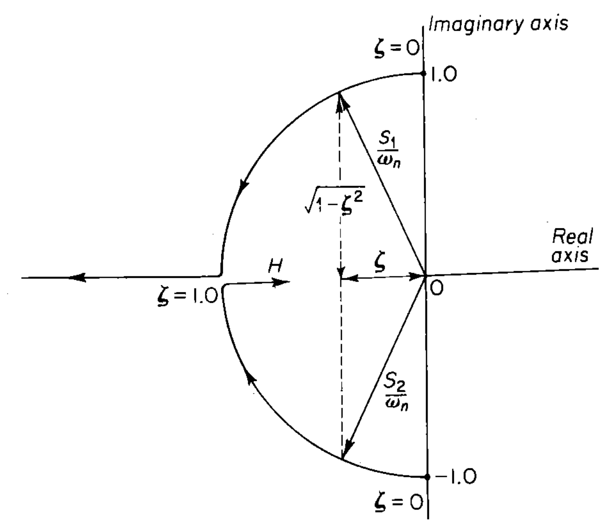

Figure 2.3-2.

converging at the point $s_{1,2}/\omega_n = -1.0$. As ζ increases beyond unity, the roots separate along the horizontal axis and remain real numbers. With this diagram in mind, we are now ready to examine the solution given by Eq. (2.3-9).

Oscillatory Motion. [$\zeta < 1.0$ (Underdamped Case).] Substituting Eq. (2.3-12) into (2.3-8), the general solution becomes

$$x = e^{-\zeta\omega_n t}\left(Ae^{i\sqrt{1-\zeta^2}\,\omega_n t} + Be^{-i\sqrt{1-\zeta^2}\,\omega_n t}\right) \qquad (2.3\text{-}13)$$

The above equation can also be written in either of the following two forms

$$x = Xe^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right) \qquad (2.3\text{-}14)$$

$$= e^{-\zeta\omega_n t}\left(C_1 \sin\sqrt{1-\zeta^2}\,\omega_n t + C_2 \cos\sqrt{1-\zeta^2}\,\omega_n t\right) \qquad (2.3\text{-}15)$$

where the arbitrary constants X, ϕ, or C_1, C_2 are determined from initial conditions. With initial conditions $x(0)$ and $\dot{x}(0)$, Eq. (2.3-15) can be shown to reduce to

$$x = e^{-\zeta\omega_n t}\left(\frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_n\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t + x(0)\cos\sqrt{1-\zeta^2}\,\omega_n t\right) \qquad (2.3\text{-}16)$$

The equation indicates that the *frequency of damped oscillation* is equal to

$$\omega_d = \frac{2\pi}{\tau_d} = \omega_n\sqrt{1-\zeta^2} \qquad (2.3\text{-}17)$$

Figure 2.3-3 shows the general nature of the oscillatory motion.

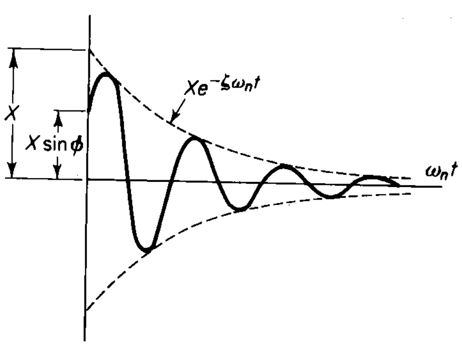

Figure 2.3-3. Damped oscillation $\zeta < 1.0$.

Nonoscillatory Motion. [$\zeta > 1.0$ (Overdamped Case).] As ζ exceeds unity, the two roots remain on the real axis of Fig. 2.3-2 and separate, one increasing and the other decreasing. The general solution then becomes

$$x = Ae^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + Be^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \qquad (2.3\text{-}18)$$

where

$$A = \frac{\dot{x}(0) + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n x(0)}{2\omega_n\sqrt{\zeta^2-1}}$$

and

$$B = \frac{-\dot{x}(0) - \left(\zeta - \sqrt{\zeta^2-1}\right)\omega_n x(0)}{2\omega_n\sqrt{\zeta^2-1}}$$

The motion is an exponentially decreasing function of time as shown in Fig. 2.3-4, and is referred to as *aperiodic*.

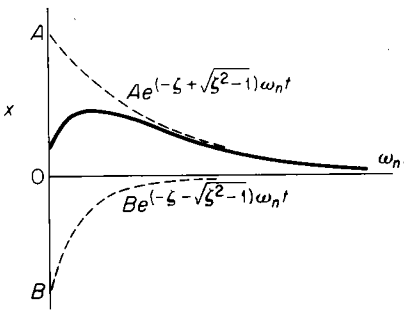

Figure 2.3-4. Aperiodic motion $\zeta > 1.0$.

Critically Damped Motion. [$\zeta = 1.0$] For $\zeta = 1$, we obtain a double root $s_1 = s_2 = -\omega_n$, and the two terms of Eq. (2.3-8) combine to form a single term

$$x = (A + B)e^{-\omega_n t} = Ce^{-\omega_n t}$$

which is lacking in the number of constants required to satisfy the two initial conditions. The solution for the initial conditions $x(0)$ and $\dot{x}(0)$ can be found from Eq. (2.3-16) by letting $\zeta \to 1$

$$x = e^{-\omega_n t}\{[\dot{x}(0) + \omega_n x(0)]t + x(0)\} \qquad (2.3\text{-}19)$$

Figure 2.3-5 shows three types of response with initial displacement $x(0)$. The moving parts of many electrical meters and instruments are critically damped to avoid overshoot and oscillation.