$$\tau_b = \frac{V\,Q}{I\,t}$$

(5.31)

where V is the applied shear, obtained using unfactored loads when calculating the elastic stresses, Q is the first moment of area about the centroid of the cross-section, I is the moment of inertia about the bending axis, and t is the thickness of the plate element of the cross-section in which the shear stress is calculated.

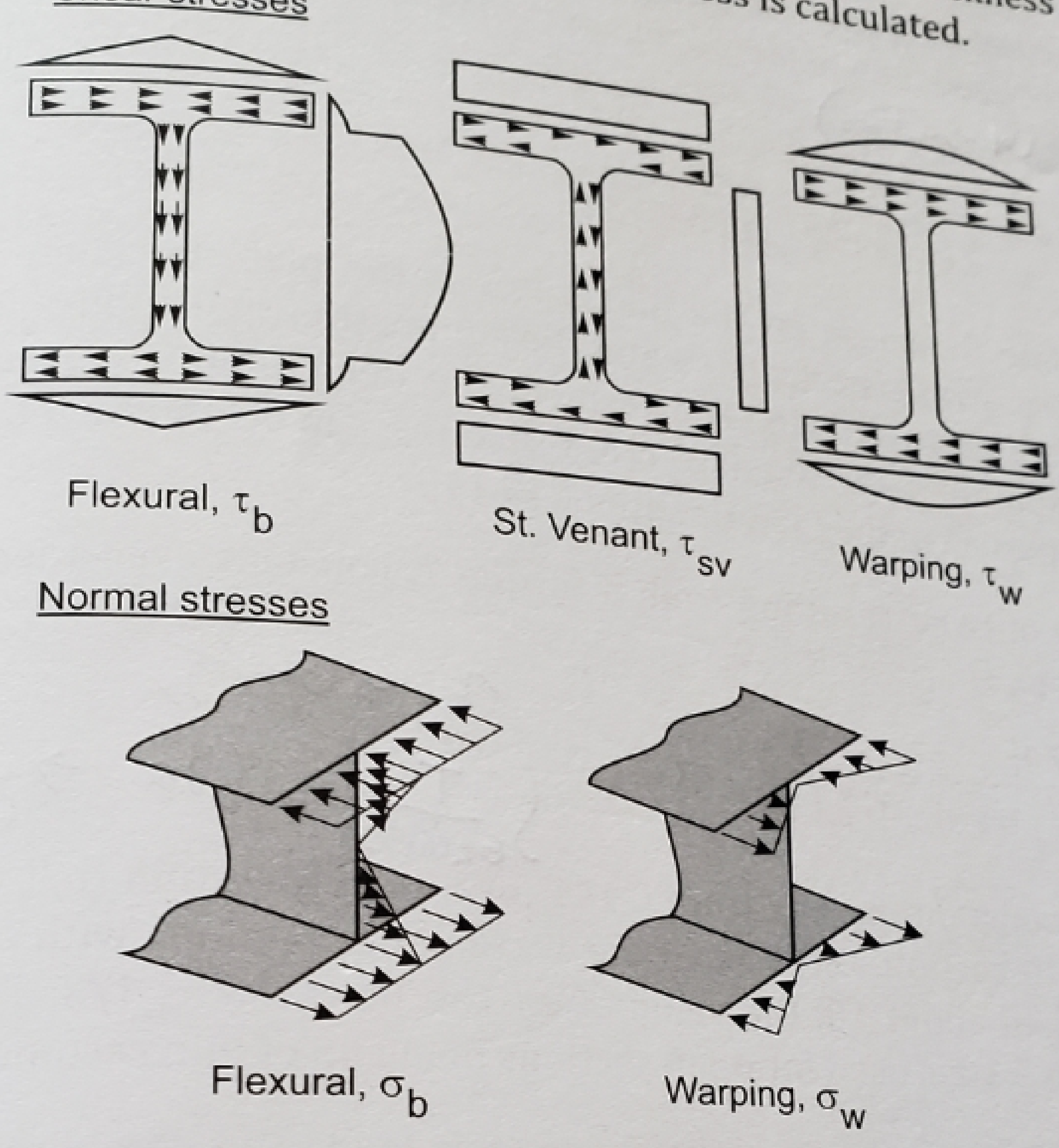


Figure 5.17 – Stresses in a W-shape due to Combined Bending and Torsion

The shear stress resulting from pure torsion, the St. Venant shear stress, τ_{SV}, which is the only type present if a member is in pure torsion (no warping restraint at either end and constant torque over the length of the member), varies linearly through the wall thickness and the maximum value on the wall surface is obtained as:

$$(\tau_{sv})_{max} = G\,t\,\frac{d\theta}{dx} \qquad (5.32)$$

where G is the shear modulus of elasticity, t is the wall thickness, and $d\theta / dx$ is the rate of change of the angle of twist along the length of the beam. For a rectangular cross-section of width b and thickness t, it can be demonstrated that the maximum shear stress is related to the internal torque, *T*, using the following relationship [5.3]:

$$(\tau_{sv})_{max} = \frac{T}{c_1\,b\,t^2} \qquad (5.33)$$

where c_1 is a constant va... very thin sections ($b/t = \infty$). For ... to 1/3 for ... where $b/t \geq 5$, the constant c_1 can be obtained from the following:

$$c_1 = \frac{1 - 0.63\, t/b}{3} \tag{5.34}$$

For cross-sections with b/t exceeding 10, the value of c_1 is commonly taken as 1/3. For open cross-sections made of thin rectangular sections, Equation 5.32 can be rewritten as: (W-sections)

$$(\tau_{sv})_{max} = \frac{T\,t}{\Sigma(c_1\, b\, t^3)} = \frac{T\,t}{\Sigma(b\, t^3/3)} \tag{5.35}$$

where the plate thickness on the numerator is the plate thickness in which the maximum shear stress is calculated and the summation sign applies to all the plate elements of the cross-section. The term in the denominator corresponds to the torsional constant and, for standard cross-sections, can be obtained from the tables of section properties in the CISC Handbook [5.14].

The angle of twist resulting from a torque T, constant over a length L, can be obtained from [5.3]:

$$\theta = \frac{T\,L}{G\,\Sigma(c_2\, b\, t^3)} = \frac{T\,L}{G\,\Sigma(b\, t^3/3)} \tag{5.36}$$

where c_2 is a constant varying from 0.141 for a square section (b/t = 1) to 1/3 for an infinitely thin section. Once again, for rectangular sections with an aspect ratio (b/t) greater than about 10, the value of c_2 can be taken as 1/3. The summation sign in Equation 5.36 is used for open sections made up of thin rectangular sections.

Pure torsion never occurs if the member is loaded in combined bending and torsion. The restrained warping, resulting either from the support boundary conditions or from non-uniform torsional moment along the span length, introduces flexural shear stresses and normal stresses in the flanges of the section. The resulting warping shear, τ_w, and warping normal stresses, σ_w, in the flanges are obtained as

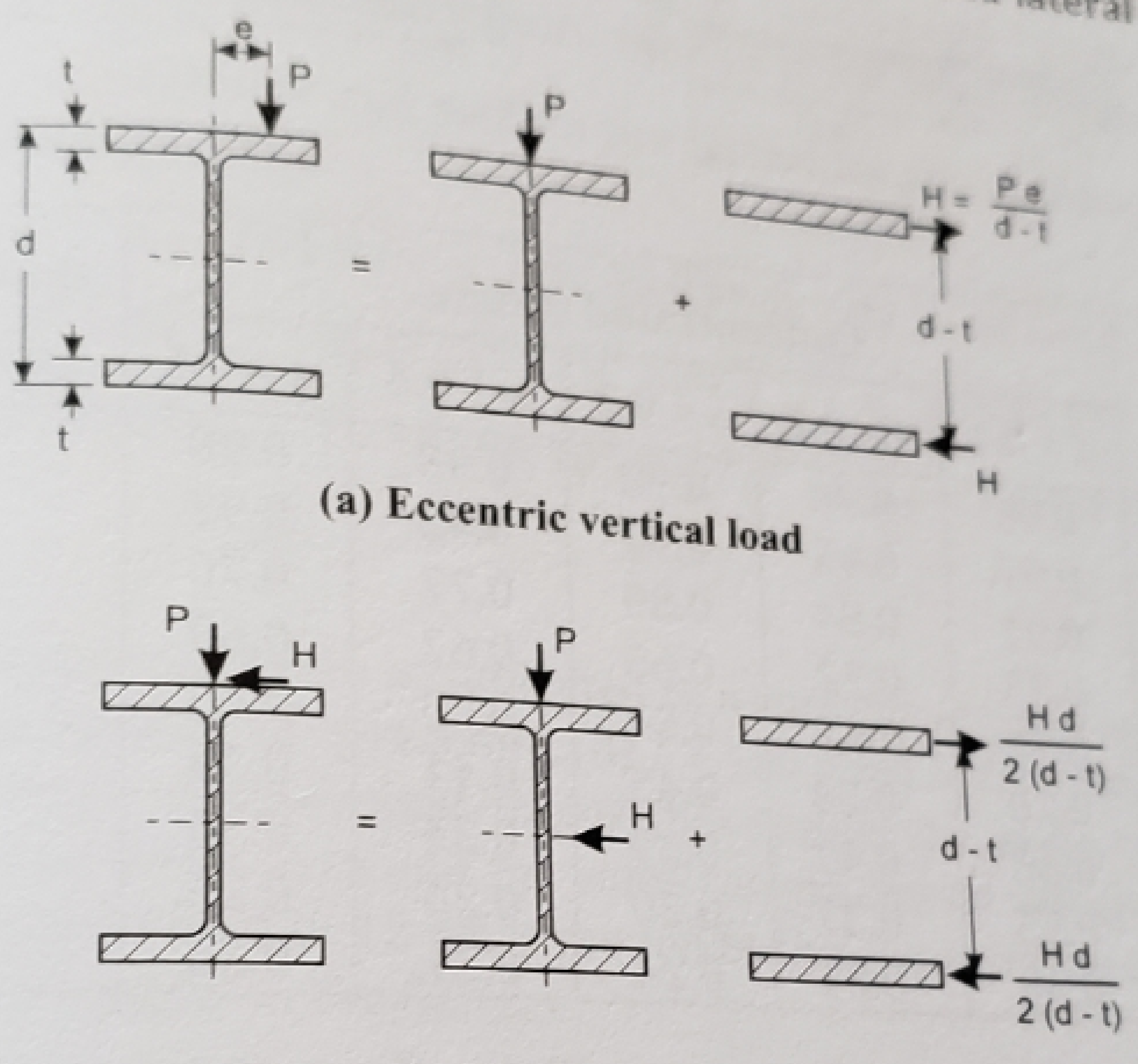
$$\tau_w = -E\,S_w \frac{d^3\theta}{dx^3} \tag{5.37}$$

$$\sigma_w = -E\,W_n \frac{d^2\theta}{dx^2} \tag{5.38}$$

where S_w is the warping statical moment for the cross-section ($= b^2 h/16$ for an I-shape section) and W_n is the normalized unit warping for the cross section ($= b\,h/4$ for an I-shaped section).

The angle of twist, θ, is obtained from the solution of the differential equation obtained when the external couple is equilibrated by the internal force couple describing the resisting torque of a beam. The angle of twist, and hence the stresses from the general solution, are determined from the support conditions at the ends of the member. The exact solution of this problem is classic and all the details can be found elsewhere [5.22, 5.23, 5.24].

An approximate solution to the torsion problem is to consider the applied torsional moment as a couple acting in the plane of the two flanges. The flanges are then treated as loaded in their plane, thus transforming the torsion problem into a beam flexure problem. The stresses in the flanges are computed from beam theory, assuming that the flanges act as rectangular beams bending about their strong axis. If torsion results from an eccentric vertical force, the flanges are subjected to equal lateral forces H, which produce lateral bending of the flanges. This is illustrated in Figure 5.18(a). When torsion results from an eccentric lateral load, a statically equivalent system consists of a biaxial bending component and lateral forces in the flanges, Figure 5.18(b).



(a) Eccentric vertical load

(b) Vertical and horizontal loads

Figure 5.18 – Flexure Analogy

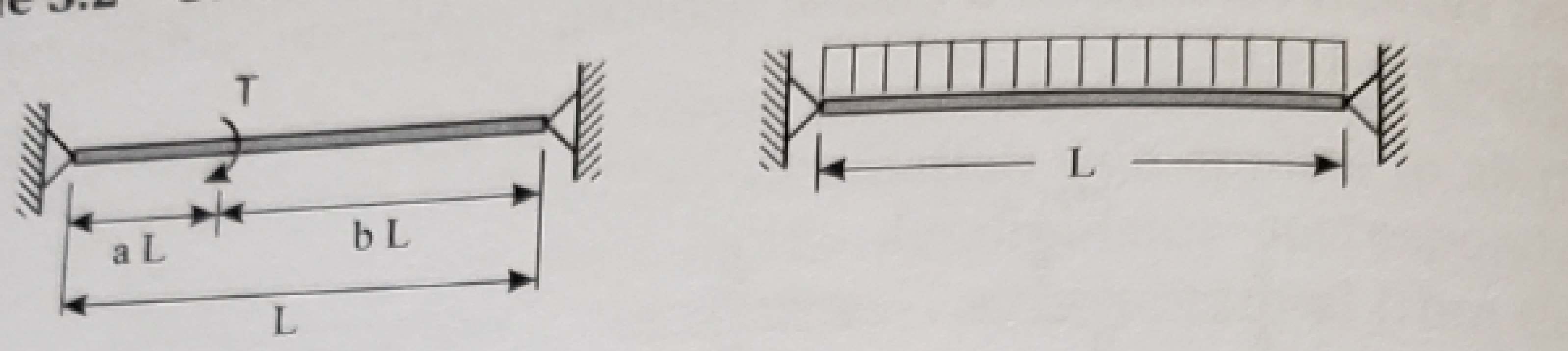
Because the simple flexure analogy assumes that the applied torque is entirely carried by warping, i.e., the pure torsion contribution is negligible, the flexure analogy is conservative for the vast majority of beams with simple supports, i.e., it overestimates the bending normal stresses introduced in the flange plates. For sections with a large ratio of pure torsion stiffness (G J) to warping torsion stiffness (EC_w), the pure torsion component can be important compared to the warping component. The simplifying assumption of the simple flexure analogy can therefore greatly overestimate the stresses due to warping. In order to circumvent this problem, Lin [5.25] proposed a modification to the flexure analogy whereby a correction factor, β, is applied to reduce the effect of the equivalent lateral loads. The resulting reduced stresses become very close to the values calculated using the exact elastic method. The value of this correction factor depends on the support conditions, the nature of the load causing the torsion, the length of the member, and the cross-section properties. Some values of the correction factor are presented in

Table 5.2 for a beam simply supported in torsion at both extremities and for two load cases: a concentrated torque, and a uniformly distributed torque.

The values presented in the table represent the maximum load effect, namely, at the point of the applied torque for concentrated torque cases, and at midspan (a = 0.5) for the uniform torque case. The factor μ in Table 5.2 represents the ratio of the pure torsion stiffness to the warping torsion stiffness and is given as

$$\mu = \sqrt{\frac{G\,J}{E\,C_w}} \tag{5.39}$$

Table 5.2 – Correction factor β for the modified flexure analogy



μ L	Concentrated torque					Uniform Torque (β at midspan)
	a = 0.1	a = 0.2	a = 0.3	a = 0.4	a = 0.5	
0.5	0.99	0.99	0.98	0.98	0.98	0.97
1.0	0.97	0.95	0.94	0.93	0.92	0.91
2.0	0.91	0.84	0.80	0.77	0.76	0.70
3.0	0.83	0.72	0.65	0.62	0.60	0.51
4.0	0.76	0.62	0.54	0.50	0.48	0.37
5.0	0.70	0.54	0.45	0.41	0.39	0.27
6.0	0.65	0.47	0.39	0.34	0.33	0.20
8.0	0.55	0.37	0.30	0.26	0.25	0.12
10.0	0.48	0.31	0.24	0.21	0.20	0.08

As expected, when the factor μ is small, i.e., the warping torsion stiffness term $E\,C_w$ is large compared to the pure torsion stiffness term $G\,J$, a larger portion of the torque is carried by warping and the error in the simplified flexural analogy is small. This is reflected by a value of β approaching 1.0 (i.e., all the torque is carried by warping) as the value μ L decreases.

Example 5.9

Given

The simply supported beam shown in Figure 5.19 consists of a W460× 106 with a span length of 7400 mm. It is loaded at midspan with a concentrated gravity load of 90 kN and a concentrated lateral load of 12 kN applied 150 mm above the top flange of the section. The ends of the beam are simply supported with respect to torsional restraint (i.e., θ = 0 at the ends). Compute the stresses due to bending and torsion. The steel is of grade ASTM A992.

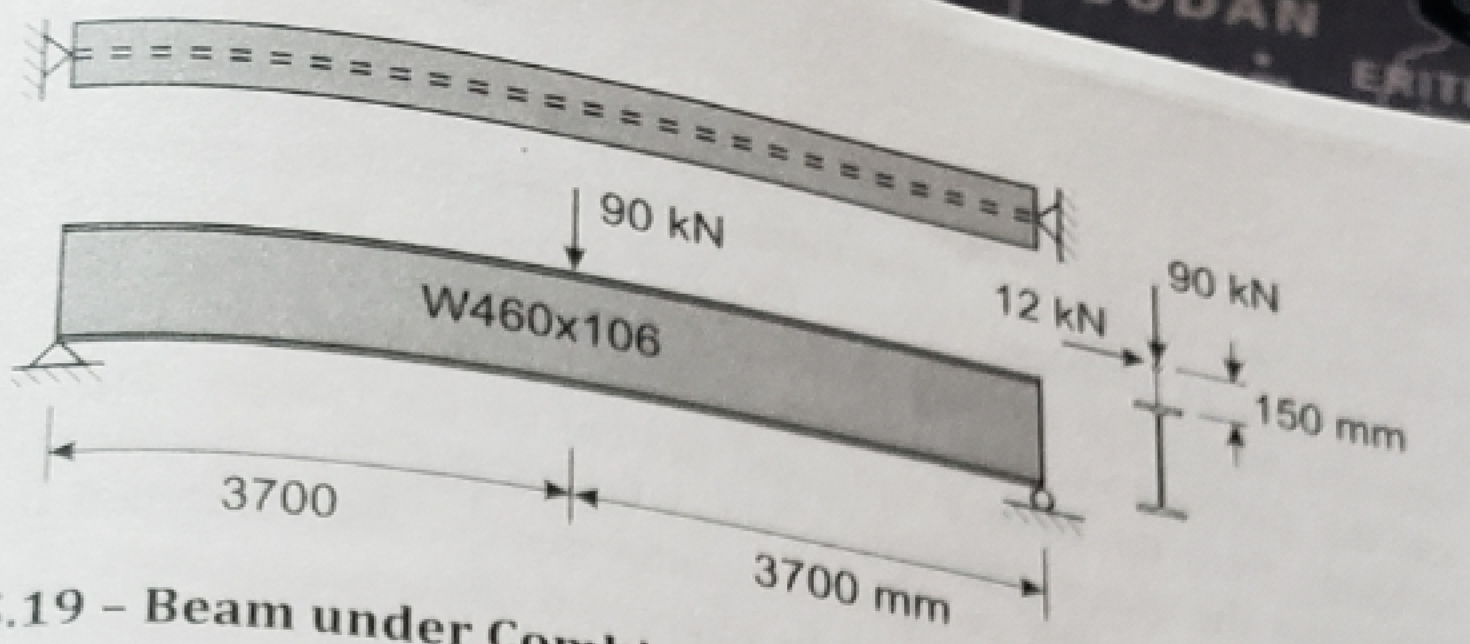


Figure 5.19 – Beam under Combined Bending and Torsion — Example 5.9

Solution

Two different approaches will be used for the calculations: (1) using the flexure analogy, and (2) using the modified flexure analogy proposed in Reference [5.25].

The required dimensions and section properties for the W460× 106 are listed in the CISC Handbook [5.14]:

$d = 469$ mm; $b = 194$ mm; $t = 20.6$ mm; $w = 12.6$ mm;

$S_x = 2080 \times 10^3$ mm^3; $S_y = 259 \times 10^3$ mm^3; $I_x = 488 \times 10^6$ mm^4

$I_y = 25.1 \times 10^6$ mm^4; $J = 1460 \times 10^3$ mm^4; $C_w = 1260 \times 10^9$ mm^6

Flexural analogy

The simply supported beam is subjected to a concentrated torque caused by the 12 kN lateral load applied at 384.5 mm (d/2 + 150 mm) above the shear centre of the section. A system of equivalent forces that create the same bending moment about the strong and weak axes and the same torque can be calculated as shown in Figure 5.20. The lateral load of 12 kN is moved to the shear centre and the torque is replaced by a force couple on the flanges.

The problem consequently becomes a combined stress problem, where the first component consists of the beam subjected to the gravity and lateral loads applied at the shear centre, causing bending about the strong and the weak axes, respectively, and the second component consists of the flanges subjected to the equivalent force couple. Both normal and shear stresses can be calculated using simple beam theory.

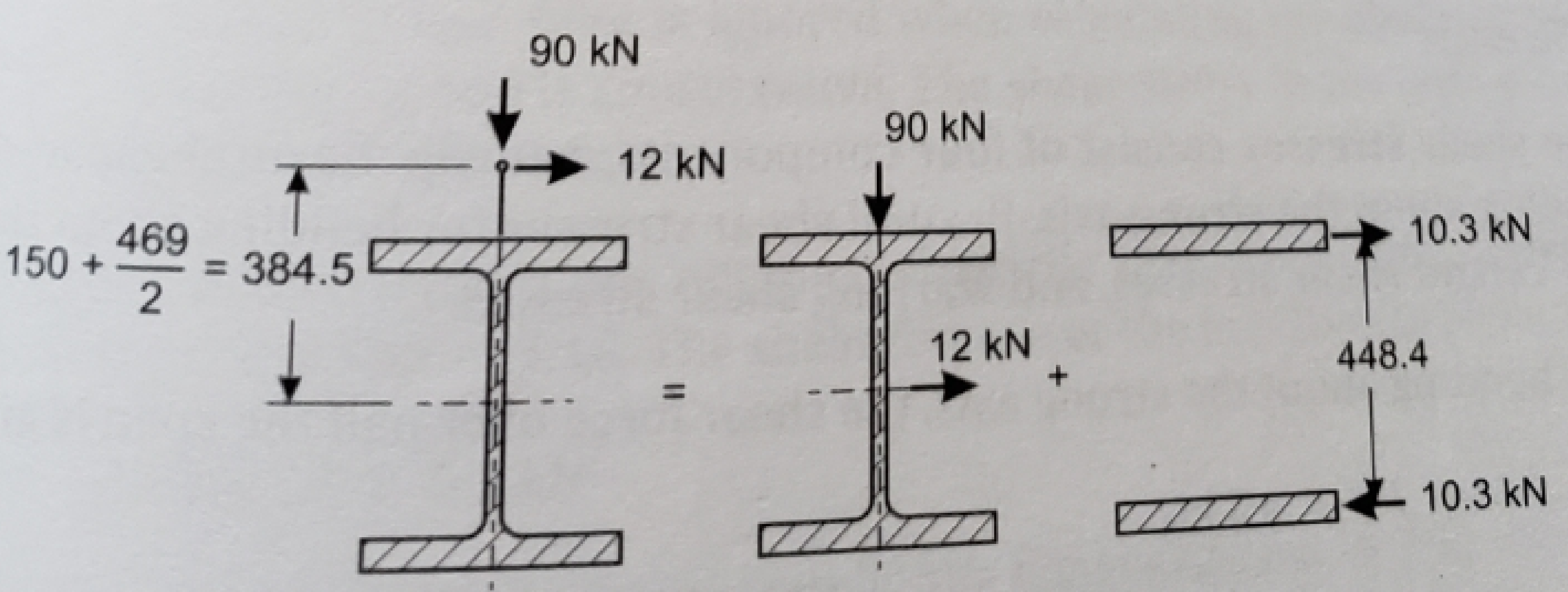


Figure 5.20 – Equivalent Forces on Beam Cross-Section

Normal Stresses

Normal stresses reach their maximum where the bending moments reach their maximum, namely, at midspan. Three flexural moments are obtained, namely, the bending moment on the full cross-section caused by the gravity load, M_x, the bending moment on the full cross-section caused by the lateral load, M_y, and the bending moment resulting from the force couple and carried by the flanges, M_{fl}. The first two moments represent the biaxial bending effect, whereas the third moment represents the warping effect. Assuming that conditions remain elastic, the combined normal stress, which is maximum at the flange tips, can be expressed as:

$$\sigma = \frac{M_x}{S_x} + \frac{M_y}{S_y} + \frac{M_{fl}}{S_y/2}$$

where $S_y/2$ is approximately equal to the elastic section modulus of one flange about the axis of symmetry of the cross-section, which can also be taken as $t\,b^2/6$.
Since the beam is simply supported at its ends,

$$M_x = \frac{PL}{4} = \frac{90\ \text{kN} \times 7.4\ \text{m}}{4} = 166.5\ \text{kN}\cdot\text{m}$$

$$M_y = \frac{12\ \text{kN} \times 7.4\ \text{m}}{4} = 22.2\ \text{kN}\cdot\text{m}$$

$$M_{fl} = \frac{10.3\ \text{kN} \times 7.4\ \text{m}}{4} = 19.1\ \text{kN}\cdot\text{m}$$

Applying the above equation for combined normal stresses, we obtain

$$\sigma = \frac{166.5\times10^6\ \text{N}\cdot\text{mm}}{2080\times10^3\ \text{mm}^3} + \frac{22.2\times10^6\ \text{N}\cdot\text{mm}}{259\times10^3\ \text{mm}^3} + \frac{19.1\times10^6\ \text{N}\cdot\text{mm}}{259\times10^3\ \text{mm}^3/2}$$

$$\sigma = 80.1\ \text{MPa} + 85.7\ \text{MPa} + 147.5\ \text{MPa} = 313\ \text{MPa}\ \text{(Compression)}$$

A summary of the calculated normal stresses is presented in Figure 5.21. The calculated warping stress, 147.5 MPa, represents 47 percent of the total combined normal stress. Since the maximum normal stress, 313 MPa, is less than the yield strength of the material, 345 MPa, the section is considered adequate under service loads.

Shear Stresses

The shear stresses consist of four components, namely, flexural shear stresses for bending about the strong axis, flexural shear stresses for bending about the weak axis, St. Venant shear stresses, and warping shear stresses.

For bending about the strong axis, the shear force over half the span is given as:

$$V = 90\ \text{kN}/2 = 45\ \text{kN}$$

The resulting shear stress in the flange is maximum at the flange-to-web junction and is obtained as

$$\tau_{bx\ flange} = \frac{V\,Q}{I\,t} = \frac{45\,000\ N \times 448 \times 10^3\ mm^3}{488 \times 10^6\ mm^4 \times 20.6\ mm} = 2.0\ MPa$$

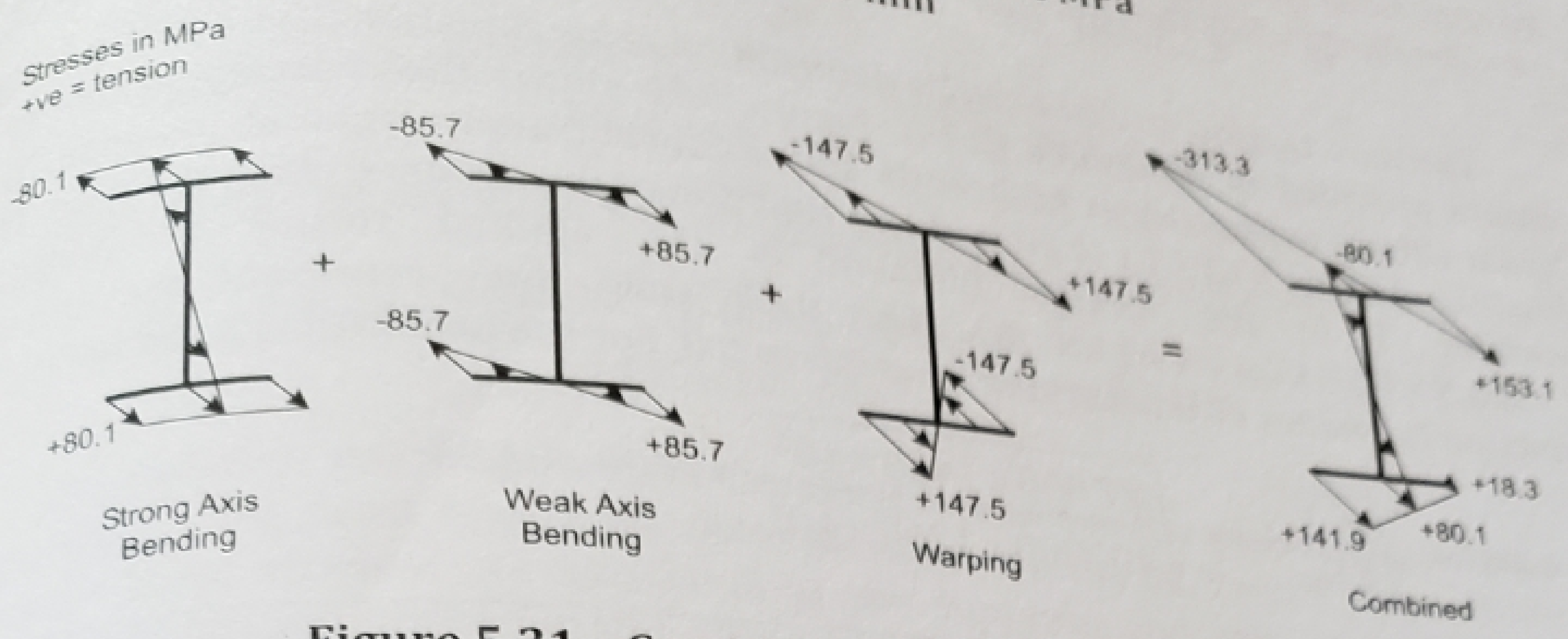


Figure 5.21 – Combination of Normal Stresses

It should be noted that Q, the first moment of area, in the above calculation was conservatively taken as the moment of half the flange about the strong axis of the cross-section. In reality, the flexural shear stress in the flanges reaches its maximum value in the fillet between the flange and the web and the increased thickness of the flange in the fillet should be used in the calculation.

The flexural shear stress in the web, which is maximum at the centroid of the cross-section, is given as

$$\tau_{bx\ web} = \frac{V\,Q}{I\,t} = \frac{45\,000\ N \times 1.186 \times 10^6\ mm^3}{488 \times 10^6\ mm^4 \times 12.6\ mm} = 8.7\ MPa$$

For bending about the weak axis, the shear force over half the span and the resulting maximum shear stress in the flanges are given as:

$V = 12\ kN / 2 = 6\ kN$

$$\tau_{by\ flange} = \frac{V\,Q}{I\,t} = \frac{6000\ N \times 96.5 \times 10^3\ mm^3}{25.1 \times 10^6\ mm^4 \times 20.6\ mm} = 1.1\ MPa$$

The value of Q was calculated in the flange at the face of the web. Again, the effect of the flange to web fillet is ignored when calculating the shear stress, which simplifies the problem and is conservative. The shear stress in the web is negligible due to bending about the weak axis.

Warping shear stresses are obtained from the equivalent shear forces in the flanges as shown in Figure 5.18. The shear force over the half span is given as:

$V = 10.3\ kN / 2 = 5.2\ kN$

The resulting maximum shear stress in the flanges caused by this shear force is given as:

$$\tau_w = \frac{VQ}{It} = \frac{5200\,N \times 96.5 \times 10^3\,mm^3}{12.5 \times 10^6\,mm^4 \times 20.6\,mm} = 2.0\,MPa$$

The pure torsion stresses can be determined using Equation 5.31 and the design charts provided in Reference [5.24]. Figure 5.22 shows the chart applicable to a beam simply supported at both ends and a concentrated torque at midspan. The function $(d\phi/dx)(GJ/T)$ is plotted as a function of the distance along the span. The variable T in the torsion function is the applied torque at midspan $(= 12\,kN \times 0.3845\,m = 4.61\,kN \cdot m)$. The dark solid lines represent the torsion function for values of μL varying from 0.5 to 6.0. For the beam considered here,

$$\mu L = \sqrt{\frac{GJ}{EC_w}}\,L = \sqrt{\frac{77\,000 \times 1460 \times 10^3}{200000 \times 1260 \times 10^9}} \times 7400 = 4.94$$

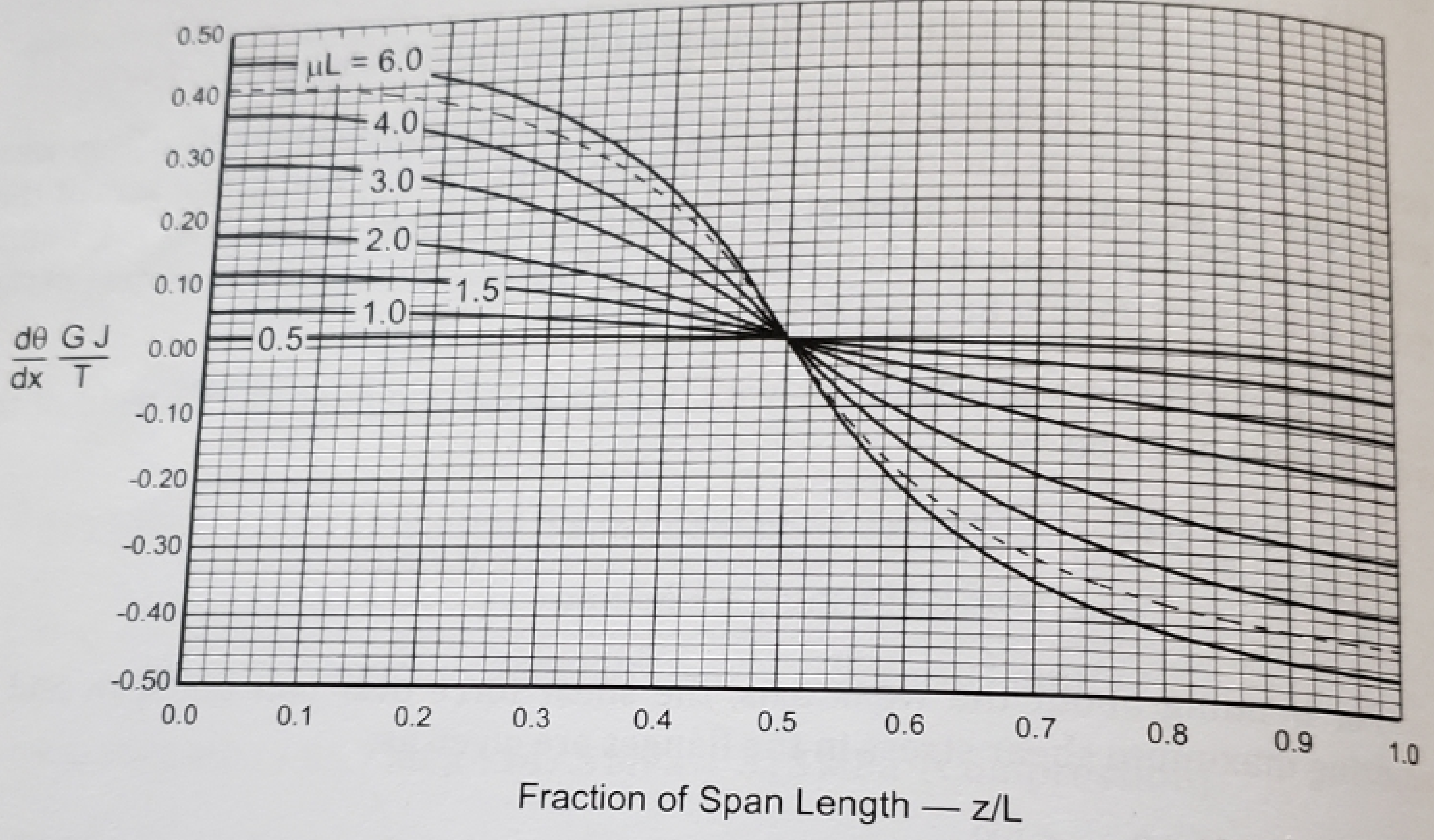


Figure 5.22 – Torsion Function for Concentrated Torque at Midspan

The dotted line in Figure 5.22 represents the torsion function for $\mu L = 4.94$. The torsion function is maximum at the end supports and zero at midspan. From Figure 5.22 the maximum value of the torsion function is 0.41, from which:

$$\frac{d\theta}{dx} = 0.41\frac{T}{GJ}$$

Substituting into Equation 5.32, we obtain

$$(\tau_{SV})_{max} = Gt\frac{d\theta}{dx} = 0.41\frac{Tt}{J}$$

Using Equation 5.35, one would use the internal torque, which, for the simply supported beam illustrated in Figure 5.18, is half of the applied torque and it can be seen that Equation 5.35 yields a constant 0.50 rather than 0.41 as obtained above.

The pure torsion stress is a function of the plate thickness, t, and will therefore be different in the flanges and in the web. The maximum St. Venant shear stress, τ_{SV}, in the flanges is

$$(\tau_{SV})_{max} = 0.41 \times \frac{4.61 \times 10^6 \text{ N} \cdot \text{mm} \times 20.6 \text{ mm}}{1460 \times 10^3 \text{ mm}^4} = 26.7 \text{ MPa}$$

Similarly, the maximum St. Venant shear stress in the web is

$$(\tau_{SV})_{max} = 0.41 \times \frac{4.61 \times 10^6 \text{ N} \cdot \text{mmm} \times 12.6 \text{ mm}}{1460 \times 10^3 \text{ mm}^4} = 16.3 \text{ MPa}$$

The variation and magnitude of the shear stresses at the supports are shown in Figure 5.23. The most significant shear stresses are the St. Venant shear stresses. Because these stresses are maximum at the supports and zero at midspan, they cannot simply be added to the normal stresses calculated above. The normal stresses were calculated at midspan, where they are maximum. Since the shear stresses are usually small and because the maximum shear stresses do not occur at the same cross-section as the maximum normal stresses, it is common practice to ignore the shear stresses when checking the cross-section for yielding.

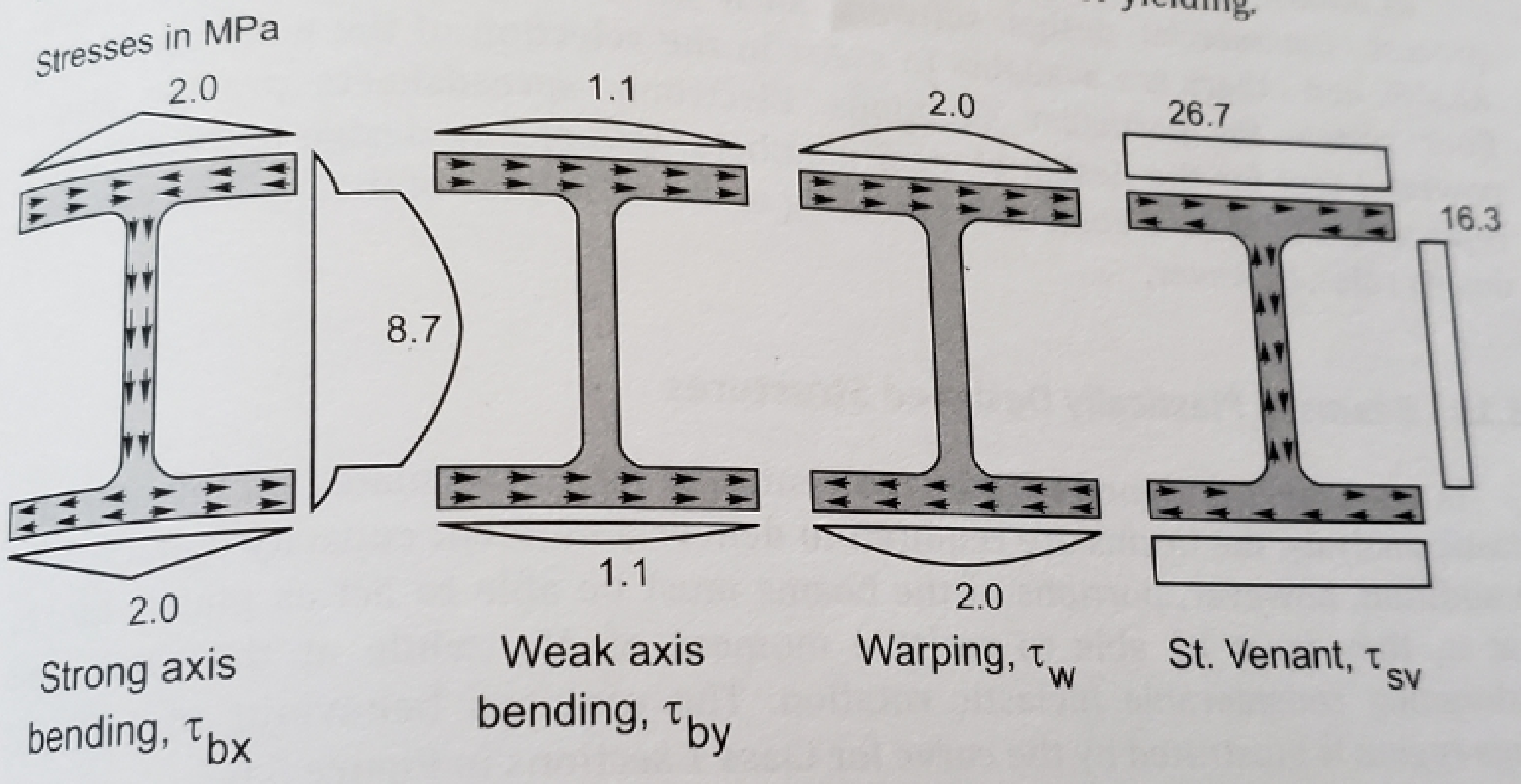


Figure 5.23 – Shear Stresses

Modified flexure analogy

The calculation of the warping normal stress given above is based on the assumption that torsion is resisted entirely by warping. The method of Reference [5.26] proposes a correction factor β based on the ratio of the pure torsion to warping stiffness, μL, given as

$$\mu L = \sqrt{\frac{GJ}{EC_w}}\, L = \sqrt{\frac{77\,000 \times 1460 \times 10^3}{200000 \times 1260 \times 10^9}} \times 7400 = 4.943$$

Using this calculated stiffness ratio and Table 5.2, the correction factor β is approximately 0.4. The adjusted warping normal stress thus becomes 0.4×147.5 MPa