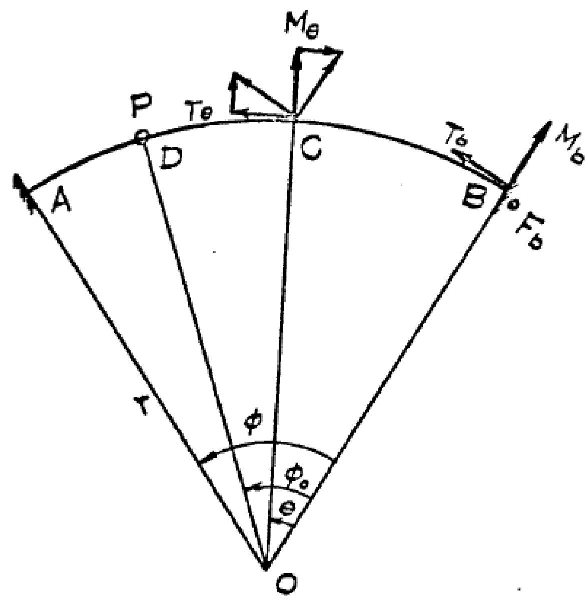
1) **Horizontally curved beam fixed at both ends and subjected to concentrated load**



As shown in Figure, a concentrated load acts at the point D with an angular distance Φ from support B. M_θ and T_θ represent the bending moment and torsional moment at any section C with an angular distance θ from B.

For portion BD ($0 \leq \theta \leq \Phi_0$)

$$M_\theta = M_b \cos\theta + T_b \sin\theta - F_b r \sin\theta \tag{1}$$

$$T_\theta = -M_b \sin\theta + T_b \cos\theta + ¿ F_b r ¿¿ \tag{2}$$

For portion DA ($\Phi_0 \leq \theta \leq \Phi$)

$$M_\theta = M_b \cos\theta + T_b \sin\theta - F_b r \sin\theta + Pr \sin\left(\theta - Ø_0\right) \tag{3}$$

$$T_\theta = -M_b \sin\theta + T_b \cos\theta + ¿ F_b r ¿¿ \tag{4}$$

Applying Castigliano's theorem with U representing the strain energy

$$U = \int \frac{M_\theta^2}{2EI} r d\theta + \int \frac{T_\theta^2}{2GJ} r d\theta \tag{5}$$

At support B

$$Ш_b \neq 0 Y_b = \Delta_b = 0$$

$$\frac{\partial U}{\partial M_b} \neq 0 \, \frac{\partial U}{\partial T_b} = \frac{\partial U}{\partial F_b} = 0 \quad (6)$$

From (5) and (6)

$$\frac{\partial U}{\partial M_b} = \frac{1}{EI} \int \frac{M_\theta . M_\theta}{M_b} r d\theta + \frac{1}{GJ} \int \frac{T_\theta . T_\theta}{M_b} r d\theta = 0 \quad (7)$$

Partially differentiating (1) and (2) by M_b

$$\frac{\partial M_\theta}{\partial M_b} = \cos\theta , \frac{\partial T_\theta}{\partial M_b} = -\sin\theta$$

Let

$$m = \frac{EI}{GJ}$$

Applying above results to (7)

$$\frac{1}{EI} \left[\int M_\theta r \cos\theta \, d\theta + m \int T_\theta r (-\sin\theta) \, d\theta \right] = 0 \quad (8)$$

Applying from (1), (2), (3) and (4)

$$\frac{r}{EI} ¿$$

$$\frac{r}{EI} ¿ = 0$$

$$\frac{r}{EI} \left[M_b \left(\frac{\emptyset}{2} + \frac{\sin 2\emptyset}{4} + m \frac{\emptyset}{2} - m \frac{\sin 2\emptyset}{4} \right) + T_b \left(\frac{-1}{4} (\cos(2\emptyset - 1)) + \frac{m}{4} (\cos(2\emptyset - 1)) \right) + F_b r \left(\frac{(1-}{4} \right. \right.$$

$$M_b [\emptyset (m+1) - \sin\emptyset \cos\emptyset (m-1)] - T_b \sin^2\emptyset (m-1) + F_b r [\sin^2\emptyset (m-1) + 2m(\cos\emptyset - 1)] = Pr \Big[\quad (9)$$

Similarly, for $\frac{\partial U}{\partial T_b} = ¿ \, 0$ we can obtain

$$-M_b [\sin^2\emptyset (m-1)] + T_b [\emptyset (m+1) + \sin\emptyset \cos\emptyset (m-1)] + F_b r [2m \sin\emptyset - \emptyset (m+1) - \sin\emptyset \cos \quad (10)$$

Similarly, for $\frac{\partial U}{\partial F_b} = ¿ \, 0$ we can obtain

$$M_b\left[\sin^2 Ø_0(m-1)+2m(\cos Ø -1)\right]+T_b\left[2m\sin Ø - Ø\ (m+1)-\sin Ø\ \cos Ø\ (m-1)\right]+F_b r\left[Ø\ (m\cdot \right. \quad (11)$$

Let

$$a_1=Ø\ (m+1)-\sin Ø\ \cos Ø\ (m-1) \quad (12)$$
$$b_1=\sin^2 Ø\ (m-1) \quad (13)$$
$$c_1=\sin^2 Ø\ (m-1)+2m(\cos Ø\ -1) \quad (14)$$
$$d_1=(m-1)\cos Ø_0\left(\sin^2 Ø\ -\sin^2 Ø_0\right)+\sin Ø_0\left(Ø\ -Ø_0\right)(m+1)-\sin Ø_0\frac{(m-1)}{2}\left(\sin 2Ø\ -\sin 2Ø_0\right. \quad (15)$$

$$a_2=b_1 \quad (16)$$
$$b_2=Ø\ (m+1)+\sin Ø\ \cos Ø\ (m-1) \quad (17)$$
$$c_2=2m\sin Ø\ -Ø\ (m+1)-\sin Ø\ \cos Ø\ (m-1) \quad (18)$$
$$d_2=-\cos Ø_0\left(Ø\ -Ø_0\right)(m+1)-\frac{1}{2}\cos Ø_0(m-1)\left(\sin 2Ø\ -\sin 2Ø_0\right)-(m-1)\sin Ø_0\left(\sin^2 Ø\ -\sin^2\right. \quad (19)$$

$$a_3=c_1 \quad (20)$$
$$b_3=c_2 \quad (21)$$
$$c_3=Ø\ (m+1)+\sin Ø\ \cos Ø\ (m-1)-4m\sin Ø\ +2mØ \quad (22)$$
$$d_3=\left[\frac{(m-1)}{2}\cos Ø_0\left(\sin 2Ø\ -\sin 2Ø_0\right)+\left(Ø\ -Ø_0\right)\left(\cos Ø_0(m+1)+2m\right)+(m-1)\sin Ø_0\left(\sin^2 Ø\ -\right.\right. \quad (23)$$

Then the equations can be simplified into following simultaneous equations

$$a_1 M_b-b_1 T_b+c_1 F_b r=d_1 Pr \quad (24)$$

$$-b_1 M_b+b_2 T_b+c_2 F_b r=d_2 Pr \quad (25)$$

$$c_1 M_b+c_2 T_b+c_3 F_b r=d_3 Pr \quad (26)$$

Just need to solve above two simultaneous equations now.

Placing above equation system in matrix form and carrying out operations for matrices the following result can be obtained.

$$|A|=\begin{vmatrix} a_1 & -b_1 & c_1 \\ -b_1 & b_2 & c_2 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$M_b=\frac{Pr}{|A|}\begin{vmatrix} d_1 & -b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & c_2 & c_3 \end{vmatrix} \quad (27)$$

$$T_b = \frac{Pr}{|A|} \begin{vmatrix} a_1 & d_1 & c_1 \\ -b_1 & d_2 & c_2 \\ c_1 & d_3 & c_3 \end{vmatrix} \tag{28}$$

$$F_b = \frac{P}{|A|} \begin{vmatrix} a_1 & -b_1 & d_1 \\ -b_1 & b_2 & d_2 \\ c_1 & c_2 & d_3 \end{vmatrix} \tag{29}$$