I. <u>Transfer matrix program - General</u>

The goal is to determine the critical speed of a rotor.

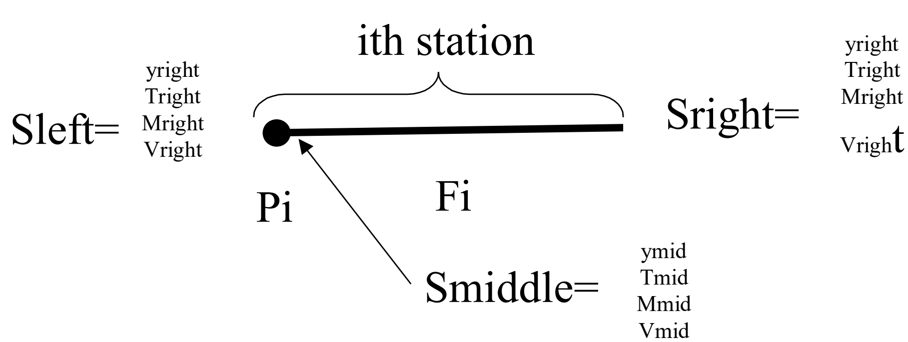
The rotor will be divided into N station numbered from left to right.

Each station is shown as a dot and a line to indicate that each station will contain some properties that can be lumped (associated with the dot and a "point matrix") and some properties that are distributed (associated with the line and a "field matrix").

At the boundary between each dot and line (and on both ends of the rotor), we can define a state vector S = [Y, T, M, V] '   where Y is transverse (vertical in this case) displacement, T (for theta) is slope (dY/dx where x is axial direction), M for moment (M=-E I dT/dx) and V for shear (V = dM/dx).

We can represent the transformation of the state variable in going from left to right side of a point in terms of a "point matrix" (P) such that when we left-multiply the left-side state by the point matrix, we get the right-side state. Similarly , we can represent the transformation of the state variable in going from left to right side of a line in terms of a "field matrix" (F).

A typical i'th element is shown below:



The equations which might be deduced:
Smiddle = Pi * Sleft
Sright = Fi* Smiddle
Sright = Fi * Pi * Sleft

If we assemble n elements as shown above, the relation between right and left side state variables would be:

Sright = Pn+1 * (Fn * Pn) * (Fn-1*Pn-1) **....(F2***P2)*(F1*P1)* Sleft

If we denote the product (Fi*Pi) as "station matrix" or "section matrix" Qi, then we can write:
Sright = Pn+1 * Qn * Qn-1 * .... Q2* Q1* Sleft

Further defining Qsystem as composite system matrix, we have
Sright = Qsystem* Sleft

In developing the point and field matrices to address a distributed system, we have to decide which properties will be handled in the field matrix and which will be handled in the point matrix.

The rotosolve program provides two different options:
"Use lumped mass calc" = true – Uses lumped mass calculation. The mass associated with a given segment is split into two and inserted into the point matrices on each end.

"Use lumped mass calc" = false – Uses distributed mass calculation. The mass associated with a given segment is split into two and inserted into the point matrices on each end.

For the lumped-mass calculation, the following field matrix is used:

$$FieldMatrixLumped = \begin{bmatrix} 1 & L & \frac{1}{2}\frac{L^2}{EI} & \frac{1}{6}\frac{L^3}{EI} \\ 0 & 1 & \frac{L}{EI} & \frac{1}{2}\frac{L^2}{EI} \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A derivation of the field matrix for the lumped-mass case is provided here: xfmatdiscrete.pdf

For the distributed-mass calculation, the following field matrix is used:

$$Fi \begin{bmatrix} \%1 & \frac{1}{2}\frac{\sin(k\,x)+\sinh(k\,x)}{k} & -\frac{1}{2}\frac{\cos(k\,x)-\cosh(k\,x)}{EI\,k^2} & \frac{1}{2}\frac{-\sin(k\,x)+\sinh(k\,x)}{EI\,k^3} \\ -\frac{1}{2}k\sin(k\,x)+\frac{1}{2}k\sinh(k\,x) & \%1 & \frac{1}{2}\frac{\sin(k\,x)+\sinh(k\,x)}{k\,EI} & -\frac{1}{2}\frac{\cos(k\,x)-\cosh(k\,x)}{EI\,k^2} \\ -\frac{1}{2}EI\,k^2\cos(k\,x)+\frac{1}{2}EI\,k^2\cosh(k\,x) & -\frac{1}{2}EI\,k\sin(k\,x)+\frac{1}{2}EI\,k\sinh(k\,x) & \%1 & \frac{1}{2}\frac{\sin(k\,x)+\sinh(k\,x)}{k} \\ \frac{1}{2}EI\,k^3\sin(k\,x)+\frac{1}{2}EI\,k^3\sinh(k\,x) & -\frac{1}{2}EI\,k^2\cos(k\,x)+\frac{1}{2}EI\,k^2\cosh(k\,x) & -\frac{1}{2}k\sin(k\,x)+\frac{1}{2}k\sinh(k\,x) & \%1 \end{bmatrix}$$

$$\%1 := \frac{1}{2}\cos(k\,x)+\frac{1}{2}\cosh(k\,x)$$

Where %1 is given at the bottom of the matrix above and k is given by:

$$K = \left(\frac{w^2 \rho \, Area}{E \, I\_}\right)^{1/4}$$

A derivation of the field matrix for the distributed-mass case is provided here:xfmatcontinuous.pdf

The point matrix then needs only address the bearing force F = -Kbrg * y = dV/dx and disk moment M = dH/dt = w^2 J dTheta/dt and is given as Pi =

$$\text{Pi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & J\,w^2 & 1 & 0 \\ -Kbrg & 0 & 0 & 1 \end{bmatrix}$$

The parameter J is assigned different values depending upon which "gryoscopic option" is selected:
Option 1 = "Critical Speed" – includes both gryoscopic disk effect (cross-coupled from the perpendicular direction) and in-plane bending disk effect. J=Ip – Id
Option 2 = "Bump test" – includes only effect of disk bending in the plane of analysis J= – Id
Option 3 = "Simple" – Disk effects excluded. J=0.

For Ip, the following expression was used:

$$\text{Ip} = \frac{1}{2}\rho \cdot L \cdot \pi (R_{outer}{}^4 - R_{inner}{}^4)$$

For Id, the following expression was used:

$$\text{Id} = \frac{1}{4}\rho \cdot L \cdot \pi (R_{outer}{}^4 - R_{inner}{}^4)$$

In fact, the complete expression for the diametral mass moment of inertia Id about a diameter passing through the axial center of a cylindrical element is given by:

$$\text{Id\_complete} = \frac{1}{4}\rho \cdot L \cdot \pi (R_{outer}{}^4 - R_{inner}{}^4) + \frac{1}{12}\rho \cdot L^3 \cdot \pi (R_{outer}{}^2 - R_{inner}{}^2)$$

$$\text{Id\_complete} = thin - disk\_term + rod\_term$$

The first term is a thin-disk inertia as if off the cylinder mass were concentrated in a thin disk. The second term is a rod inertia, as if all the cylinder mass were concentrated along the centerline of the cylinder. In my program, I used only the thin-disk term. With a large number of sections, L gets small and the second term (proportional to L^3) quickly becomes 0, so either definition would converge to the same answer if we split the sections fine enough. But for a smaller number of sections (larger L's) , I determined by trial and error that my solution matches analytical solutions much better if I discard the second term (the rod term) and use only the first term (the thin-disk term) in Id. It may be that the rod term of Id_complete is already accounted for in the Euler Bernoulli model embodied in the remaining portions of the point and field matrix.

Note that the effect embodied in option 2 (bump test mode) is commonly referred to as "rotary inertia" in textbooks covering beam theory, even though it does not involve rotation about the axis of the beam that we normally associate with rotation in the sense of rotating equipment. Draw a modeshape and imagine the beam flexing in the plane of the paper. Pick a location where the slope is steep, such as near the nodes. During free vibration under resonant undamped conditions at the mode of interest, the magnitude at that location changes sinusoidally over time, the slope of that location also changes sinusoidally. That variation in slope is the rotation which is referred to. If we attach a disk to that location, the moment required to rotate that disk back and forth (still in the plane of the paper) will lower the resonant frequency below that predicted by the Euler Bernoulli beam model.

Once the transfer matrices are known, we can translate the state variable conditions on the left side of the rotor to the conditions on the right side of the rotor by multiplying a composite matrix representing the series combination of all the point and field elements of the rotor. We apply boundary conditions to each side of the rotor, which involves setting two of the four state variables on each side to zero. The other two variables on each side are in general non-zero. So we have two zero state variables on each side and two non-zero state variables on each side. We can determine the expression for the two zero state variables on the right side as a linear combination of the variables on the left side. Only the two non-zero state variables on the left contribute. Thus we have a 2x2 sized subset of the 4x4 composite matrix which relates the non-zero state variables on the left to the zero state variables on the right. In order for the matrix multiplication of the non-zero left hand-size variables to result in a zero output on the right-hand side variables, the determinant of that 2x2 matrix must be 0.

The solution method involves searching for values of frequency that will make the determinant of that 2x2 matrix equal to 0. These are the resonant frequencies.

For each resonant frequency, we can construct the mode-shape starting with a left-side state vector. We know two elements of the left-side state vector from the boundary conditions. One of the remaining can be arbitrarily be chosen (scale factor). The last of the four can be found by applying the linear relationship between the two non-zero left-side state variables which is inherent in the 2x2 matrix equation that was solved above. Once we have the left-side state vector, we can find the state vectors for the first node to the right of that by multiplying a single point and field matrix. This procedure is repeated working across the rotor from left to right.