A look at yield-line analysis and how to use it to determine flexure limit states.

steelwise

STATED LIMITS

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TRANSVERSE FORCES got you bent out of shape? Fear not!

A new section was added (Section J10.10) to the 2016 AISC *Specification for Structural Steel Buildings* (ANSI/AISC 360, **www.aisc.org/specifications**) to address transverse forces on plate elements. One example would be an axially loaded single-plate connection to a column web or HSS wall (see Figure 1 on the next page) where flexure and shear limit states will need to be considered. And one way to approach checking the flexure limit state is to perform a yield-line analysis.

The 15th Edition AISC *Steel Construction Manual* (**www.aisc.org/manual**) provides equations in Part 9 for commonly used yield-line patterns that provide users with strengths without having to go through the additional work of deriving a solution. In fact, you can find many yield-line solutions for specific conditions provided throughout the years in AISC's quarterly *Engineering Journal* (a free download for AISC members at **www.aisc.org/ej**). Yet the concern with providing simple, easy-to-use equations is that it may be tempting to plug and chug numbers to get the job done before one has a solid understanding of what it is that they are checking. This article will discuss the basics of a yield-line analysis. In addition, go to **www.aisc.org/yieldvid** to see a video on how to use the free drawing program Google SketchUp to check yield-lines. The video may also aid you in visualizing this method of analysis.

What is a yield-line analysis?

A yield-line analysis involves the determination of a failure pattern. This requires some engineering judgment since there could be a multitude of possible failure patterns,and some of these patterns can overestimate the strength.

Once a pattern is determined, a plastic hinge is assumed to develop along the yield-lines of this failure pattern. The external work that is done by an applied force over some amount of displacement is then set equal to the internal work which is determined by the amount of rotation that occurs along the plastic hinges. The applied force (available strength) can then be determined. Note that a yield-line analysis is an upper-bound solution. That means that the correct solution will result in the lowest available strength (see Table 1 at right).

Before we demonstrate this with a simple example, please note that the following simplification will be used: For very small angles, we can take the angle, θ (unit in radians), as equal to the deflection divided by the length (see Figure 2, next page).

For a simple-span beam, it is commonly known that the maximum point load that can be applied at the midpoint is based on $M = PL/4$ where $M = F_yZ$. When the load P is such that the resulting moment reaches the plastic strength of the beam, F_yZ, a hinge will form. In the case of a simple-span beam, a single hinge at the center will result in a failure. The maximum load, P, that can be applied is equal to $4M/L$.

When performing a yield-line analysis, we compare the external work, W_{ext}, to the internal work, W_{int}. They must be equal. For a simple span beam, a load P is applied and the beam will deflect by some amount, δ (see Figure 3, page 19). The external work is equal to $P\delta$. The internal work is equal to the flexural strength of the member and the amount of rotation it undergoes. So we can say that $P\delta = M \times$ rotation. Keep in mind that for small rotations, the angle, θ, is equal to δ/Length.

Figure 3 illustrates how the commonly known equations for a simple-span beam and a fixed-fixed beam with a point load placed at midspan can be derived. A yield-line analysis is very similar to what is shown in Figure 3, except that the moment, M, which would be based

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u	W_{int}	R_n
(in.)	**(kip-in.)**	**(kip)**
1.00	131.25	131.25
1.50	95.31	95.31
2.00	78.13	78.13
2.50	68.44	68.44
3.00	62.50	62.50
3.50	58.71	58.71
4.00	56.25	56.25
4.50	54.69	54.69
5.00	53.75	53.75
5.50	53.27	53.27
6.00	53.13	53.13
6.50	53.25	53.25
7.00	53.57	53.57
7.50	54.06	54.06
8.00	54.69	54.69
8.50	55.42	55.42
9.00	56.25	56.25
9.50	57.15	57.15
10.00	58.13	58.13

Table 1. Internal work as a function of u.

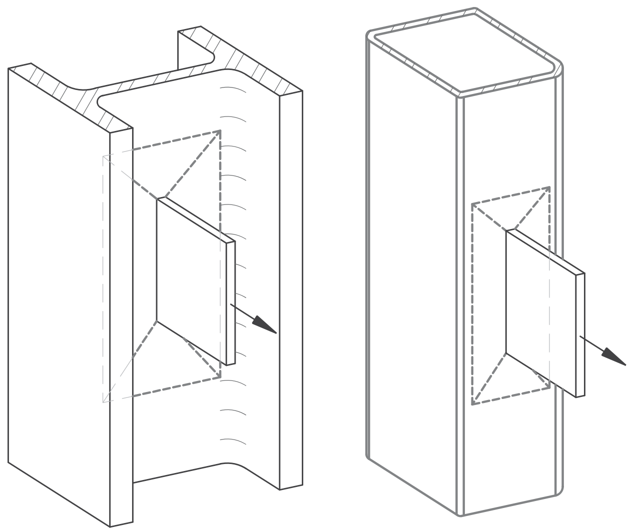

Figure 1. Yield-lines due to transverse forces on plate elements

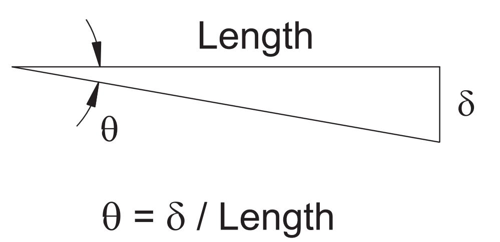

Figure 2.

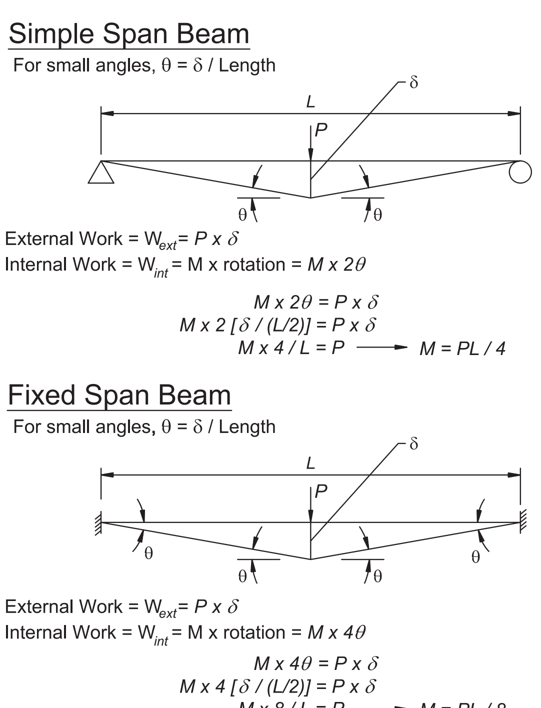

Figure 3.

on the plastic section modulus for a beam, would instead need to be calculated based on the section modulus of the plate. The plastic section modulus of the plate would depend on the length of the yield-line based on the pattern that has been assumed.

Manual Equation

As stated above, the 15th Edition *Manual* now includes equations that can be used to evaluate plate elements subjected to out-of-plane loads. This information is provided in the *Manual* to help engineers determine the strength plate elements relative to the requirements in Section J10.10 of the AISC *Specification*. Let's take a look at Equation 9-31 in the *Manual*, which can be used to evaluate out-of-plane transverse loads on column webs of wide flange sections. Note that the edges of the column web are assumed to be pinned. The variables in this equation are illustrated in Figure 4, which is recreated from Figure 9-5 in the *Manual*. Note that a variable, u, is added to Figure 4 though this dimension is not included in *Manual* Figure 9-5.

$$R_n = \frac{t_w^2 F_y}{4}\left[\frac{4\sqrt{2Tab\left(a+b\right)}+L\left(a+b\right)}{ab}\right] \qquad \text{(9-31)}$$

Pre-simplified equation

Equation 9-31 has been simplified to make it easier to use. Assuming a and b dimensions are equal, this same equation can also be written as:

$$W_{int} = M_p\theta$$

$$= \frac{t_w^2 F_y}{4}\left[\begin{array}{l} 2T\dfrac{\delta}{u}+2L\dfrac{\delta}{a} \\ +4\sqrt{a^2+u^2}\left(\dfrac{\delta\sqrt{a^2+u^2}}{au}\right)+2c\dfrac{\delta}{u}\end{array}\right]$$

The yield-line lengths in the equation above has been color coded to more easily identify with the representative yield-lines in Figure 4. The portions that have not be highlighted in the bracketed portion of the equation represent the rotation of each of those specific yield-lines.

Example

Let's solve a problem using the pre-simplified equation and compare the results to ones obtained using Equation (9-31) provided in the *Manual*.

Given: t_w = ½", F_y = 50 ksi, T = 9", a = b = 4", c = 1", L = 10", u = unknown

The variable, u, is listed as unknown. A number for u needs to be determined such that the lowest strength of the yield-line pattern is obtained. Remember that a yield-line analysis provides an upper-bound solution. The equation in the *Manual* solved for the value, u, and it is incorporated into its derivation. An Excel spreadsheet will be used here to determine the lowest value using the pre-simplified equation. Table 1 lists both the internal work and the nominal strength, R_n, which are displayed graphically in Figure 5. The internal work and nominal strength values are the same since the deformation selected, δ, is equal to 1 in.

$$W_{int} = \frac{1/2^2(50)}{4}\left[\begin{array}{l} 2\left(9\times\frac{1}{6}\right)+2\left(10\times\frac{1}{4}\right) \\ +4\sqrt{4^2+6^2}\left(\frac{1\sqrt{4^2+6^2}}{4\times 6}\right)+2\left(1\times\frac{1}{6}\right)\end{array}\right]$$

$$= 53.13 \text{ kip-in.}$$

$$W_{int} = 53.13 \text{ kip-in.} = W_{ext} = R_n\delta = R_n \times 1 \text{ in.}$$

$$R_n = 53.13 \text{ kips}$$

Solve using Equation 9-31:

$$R_n = \frac{(50 \text{ ksi})(0.5 \text{ in.}^2)}{4}\left[\frac{4\sqrt{2(9\text{in.})(4\text{in.})(4 \text{ in.})(4\text{in.}+4 \text{ in.})} + (10 \text{ in.})(4 \text{ in.} + 4\text{in.})}{(4 \text{ in.})(4 \text{ in.})}\right]$$

$$R_n = 53.13 \text{ kips}$$

As can be seen in Table 1, the lowest value matches the strength obtained from Equation 9-31. Also notice that for a wide range of u values, the strength value returned is still reasonably close to the minimum strength. This indicates that it may be possible for a designer to select a u value based on their own judgment to approximate the strength. This may be useful for conditions where closed-form yield-line equations have not been published and the demand is much lower than the approximated strength. Engineering judgment would need to be exercised with this approach. Note that assuming a u value based on a 45° distribution in the example above would have provided a nominal strength of 56.25 kips vs. 53.13 kips, a predicted strength that is about 6% higher than the correct prediction.

Though such approximations may be sufficient for many conditions encountered in practice, finding a closed-from solution that can be applied to a wide range of conditions has certain benefits and can be accomplished with some rudimentary calculus.

Geometry can also be a challenge when it comes to performing a yield-line analysis. For example, how does one determine the amount of rotation that occurs on the diagonal yield-lines in Figure 4. This can be done mathematically. The book *Design of Welded Structures* by Omer Blodgett provides a method for determining this rotation. Another possible approach is to use a 3D modeling program (like Google SketchUp).

Please keep in mind that the intent behind this article is to help gain a better understanding of the yield-line analysis method. It is important for the designer to remember that transferring load transverse to plate elements is generally not an ideal load path and should be avoided when possible. Sometimes this is not possible and, for these situations, a yield-line analysis can be used to determine that a plate element has sufficient strength. Stiffness and serviceability may also be important considerations when transferring load transverse to plate elements. One limitation of the yield line approach is that it does not produce the deformation associated with the strength meaning that it cannot be used to directly determine deflection or stiffness. ■

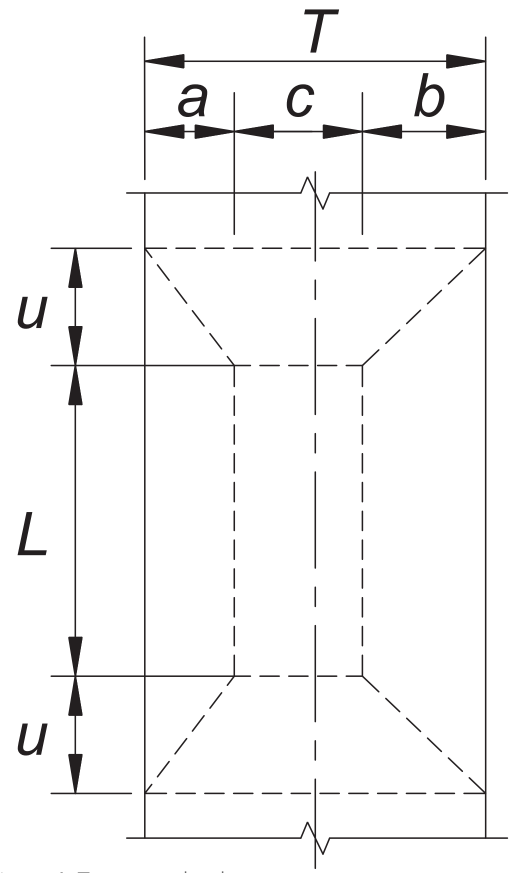

Figure 4. Transverse load.

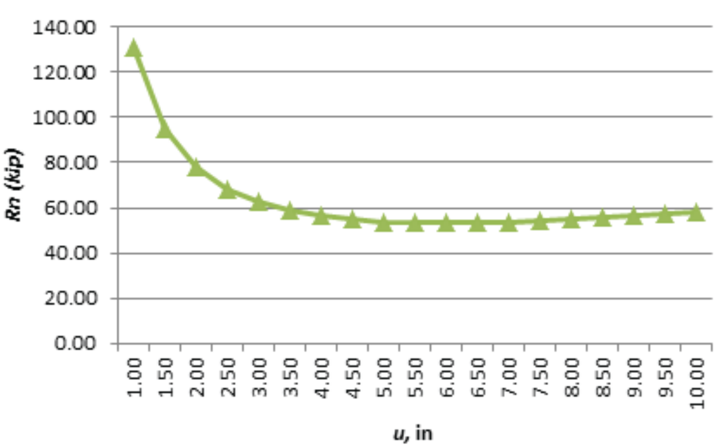

Figure 5.