Sétra
service d'Études techniques des routes et autoroutes

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Technical guide

Footbridges

Assessment of vibrational behaviour of footbridges under pedestrian loading

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RÉPUBLIQUE FRANÇAISE

ministère des Transports de l'Équipement du Tourisme et de la Mer

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Footbridges

Assessment of vibrational behaviour of footbridges under pedestrian loading

Published by the Sétra, realized within a Sétra/AFGC (French association of civil engineering) working group

This document is the translation of the work "Passerelles piétonnes - Evaluation du comportement vibratoire sous l'action des piétons", published in under the reference 0611

Association Française de Génie Civil
Régie par la loi du 1er juillet 1901
28 rue des Saints-Pères - 75007 Paris - France
téléphone : (33) 01 44 58 24 70 - télécopie : (33) 01 44 58 24 79
mél : afgc@enpc.fr – internet : http://www.afgc.asso.fr

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1. Main notations

$A(\omega)$: dynamic amplification
$[C]$: damping matrix
C_i: damping No. i in a system with n degrees of freedom (N/(m/s))
E: Young's modulus (N/m^2)
$F(t)$: dynamic excitation (N)
$[F(t)]$: dynamic load vector
F_0: amplitude of a harmonic force (N)
$[F_0]$: amplitude vector of a harmonic force (N)
$H_{r,e}(\omega)$: transfer function input e and the response r
I: inertia of a beam (m^4)
J: torsional inertia of a beam (m^4)
$[K]$: stiffness matrix
K_i: stiffness No. i in a system with n degrees of freedom (N/m)
L: length of a beam (m)
$[M]$: mass matrix
M_i: mass No. i in a system with n degrees of freedom (kg)
S: cross-sectional area of a beam (m^2)
$[X]$: vector of the degrees of freedom
C_i: generalised damping of mode i
f_0: natural frequency of a simple oscillator (Hz)
f_i: ith natural frequency of an oscillator with n degrees of freedom (Hz)
f: frequency
K_i: generalised stiffness of mode i
M_i: generalised mass of mode i
n: number of pedestrians
$[p(t)]$: modal participation vector
$[q(t)]$: modal variable vector
t: variable designating time (s)
$u(t)$, $v(t)$, $w(t)$: displacements (m)
$x(t)$: position of a simple oscillator referenced to its balance position (m)
Ω: reduced pulsation
δ: logarithmic decrement
ξ_i : ith critical damping ratio of an oscillator with n degrees of freedom (no unit)
ρ: density (kg/m^3)
$[\phi_i]$: ith eigen vector
$\phi(\omega)$: argument of $H_{r,e}$ - phase angle between entrance and exit (rad)
ω_0 : natural pulsation of a simple oscillator (rad/s): $\omega_0 = 2\pi f_0$
ω_i : ith natural pulsation of an oscillator with n degrees of freedom (rad/s): $\omega i = 2\pi fi$
ω: pulsation (rad/s)

2. Introduction

These guidelines were prepared within the framework of the Sétra/AFGC working group on "Dynamic behaviour of footbridges", led by Pascal Charles (Ile-de-France Regional Facilities Directorate and then Sétra) and Wasoodev Hoorpah (OTUA and then MIO).

These guidelines were drafted by:

Valérie BONIFACE	(RFR)
Vu BUI	(Sétra)
Philippe BRESSOLETTE	(CUST - LERMES)
Pascal CHARLES	(DREIF and then Sétra)
Xavier CESPEDES	(Setec-TPI)
François CONSIGNY	(RFR and then ADP)
Christian CREMONA	(LCPC)
Claire DELAVAUD	(DREIF)
Luc DIELEMAN	(SNCF)
Thierry DUCLOS	(Sodeteg and then Arcadis-EEGSimecsol)
Wasoodev HOORPAH	(OTUA and then MIO)
Eric JACQUELIN	(University of Lyon 1 - L2M)
Pierre MAITRE	(Socotec)
Raphael MENARD	(OTH)
Serge MONTENS	(Systra)
Philippe VION	(Sétphenomenaa)

Actions exerted by pedestrians on footbridges may result in vibrational phenomena. In general, these phenomena do not have adverse effects on structure, although the user may feel some discomfort.
These guidelines round up the state of current knowledge on dynamic behaviour of footbridges under pedestrian loading. An analytical methodology and recommendations are also proposed to guide the designer of a new footbridge when considering the resulting dynamic effects.
The methodology is based on the footbridge classification concept (as a function of traffic level) and on the required comfort level and relies on interpretation of results obtained from tests performed on the Solférino footbridge and on an experimental platform. These tests were funded by the Direction des Routes (Highway Directorate) and managed by Sétra, with the support of the scientific and technical network set up by the French "Ministère de l'Equipement, des Transports, de l'Aménagement du territoire, du Tourisme et de la Mer" and specifically by the DREIF (Division des Ouvrages d'Art et des Tunnels et Laboratoire Régional de l'Est Parisien).

This document covers the following topics:
- a description of the dynamic phenomena specific to footbridges and identification of the parameters which have an impact on the dimensioning of such structures;
- a methodology for the dynamic analysis of footbridges based on a classification according to the traffic level;
- a presentation of the practical methods for calculation of natural frequencies and modes, as well as structural response to loading;
- recommendations for the drafting of design and construction documents.

Supplementary theoretical (reminder of structural dynamics, pedestrian load modelling) and practical (damping systems, examples of recent footbridges, typical calculations) data are also provided in the guideline appendices.

Foreword

This document is based on current scientific and technical knowledge acquired both in France and abroad. French regulations do not provide any indication concerning dynamic and vibrational phenomena of footbridges whereas European rules reveal shortcomings that have been highlighted in recent works.

Two footbridges located in the centre of Paris and London, were closed shortly after inauguration: they showed hampering transverse oscillations when carrying a crowd of people; this resulted in the need for thorough investigations and studies of their behaviour under pedestrian loading. These studies involved in situ testing and confirmed the existence of a phenomenon which had been observed before but that remained unfamiliar to the scientific and technical community. This phenomenon referred to as "forced synchronisation" or "lock-in" causes the high amplitude transverse vibrations experienced by these footbridges.

In order to provide designers with the necessary information and means to avoid reproducing such incidents, it was considered useful to issue guidelines summarising the dynamic problems affecting footbridges.

These guidelines concern the normal use of footbridges as concerns the comfort criterion, and take vandalism into consideration, as concerns structural strength. These guidelines are not meant to guarantee comfort when the footbridge is the theatre of exceptional events: marathon races, demonstrations, balls, parades, inaugurations etc.

Dynamic behaviour of footbridges under wind loads is not covered in this document.

Dynamic analysis methodology is related to footbridge classification based on traffic level. This means that footbridges in an urban environment are not treated in the same way as footbridges in open country.

The part played by footbridge Owners is vital: they select the footbridge comfort criterion and this directly affects structural design. Maximum comfort does not tolerate any footbridge vibration; thus the structure will either be sturdy and probably unsightly, or slender but fitted with dampers thus making the structure more expensive and requiring sophisticated maintenance. Minimum comfort allows moderate and controlled footbridge vibrations; in this case the structure will be more slender and slim-line, i.e. more aesthetically designed in general and possibly fitted with dampers.

This document is a guide: any provisions and arrangements proposed are to be considered as advisory recommendations without any compulsory content.

3. Footbridge dynamics

This chapter draws up an inventory of current knowledge on the dynamic phenomena affecting footbridges. It also presents the various tests carried out and the studies leading to the recommendations described in the next chapter.

3.1 - Structure and footbridge dynamics

3.1.1 - General

By definition static loads are constant or barely time-variant (quasi-static) loads. On the other hand, dynamic loads are time-related and may be grouped in four categories:

- harmonic or purely sinusoidal loads;
- periodically recurrent loads integrally repeated at regular time intervals referred to as periods;
- random loads showing arbitrary variations in time, intensity, direction…;
- pulsing loads corresponding to very brief loads.

Generally speaking pedestrians loads are time-variant and may be classified in the "periodic load" category. One of the main features of the dynamic loading of pedestrians is its low intensity. Applied to very stiff and massive structures this load could hardly make them vibrate significantly. However, aesthetic, technical and technological developments lead to ever more slender and flexible structures, footbridges follow this general trend and they are currently designed and built with higher sensitivity to strain. As a consequence they more frequently require a thorough dynamic analysis.

The study of a basic model referred to as simple oscillator illustrates the dynamic analysis principles and highlights the part played by the different structural parameters involved in the process. Only the main results, directly useful to designers are mentioned here. Appendix 1 to these guidelines gives a more detailed presentation of these results and deals with their general application to complex models.

3.1.2 - Simple oscillator

The simple oscillator consists of mass m, connected to a support by a linear spring of stiffness k and a linear damper of viscosity c, impacted by an external force $F(t)$ (figure 1.1). This oscillator is supposed to move only by translation in a single direction and therefore has only one degree of freedom (herein noted "dof") defined by position $x(t)$ of its mass. Detailed calculations are provided in Appendix 1.

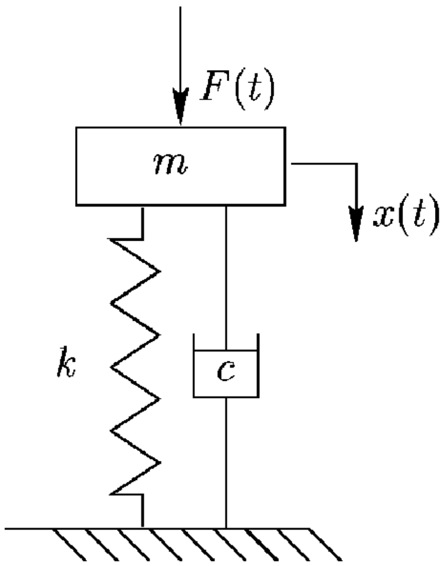


Figure 1.1: Simple oscillator

The dynamic parameters specific to this oscillator are the following:

- $\omega_0 = \sqrt{\frac{k}{m}} = 2\ \pi f_o$: natural pulsation (rad/s), f_o being the natural frequency (Hz). Since *m* is a mass, its S.I. unit is therefore expressed in kilograms.
- $\xi = \frac{c}{2\sqrt{k\ m}}$: critical damping ratio (dimensionless) or critical damping percentage. In practice, ξ has a value that is always less than 1. It should be noted that until experimental tests have been carried out, the critical damping ratio can only be assessed. Damping has various origins: it depends on materials (steel, concrete, timber) whether the concrete is cracked (reinforced concrete, prestressed concrete), the steel jointing method (bolting, welding).

The resonance phenomenon is particularly clear when the simple oscillator is excited by a harmonic (or sinusoidal) under the form $F_0 \sin(\omega t)$.

If, by definition, its static response obtained with a constant force equal to F_0 is:

$$x_{statique} = \frac{F_0}{k} = \frac{F_0\ /m}{\omega_0^2}$$

the dynamic response may be amplified by a factor $A(\Omega)$ and is equal to:

$$x_{max} = x_{statique}\, A(\Omega)$$

where $\Omega = \frac{\omega}{\omega_0}$ is the reduced (or relative) pulsation and

$$A(\Omega) = \frac{1}{\sqrt{(1-\Omega^2)^2 + 4\ \xi^2\ \Omega^2}}$$ is the dynamic amplification.

Dynamic amplification is obtained as a function of Ω and ξ. It may be represented by a set of curves parameterised by ξ. Some of these curves are provided in figure 1.2 for a few specific values of the critical damping ratio. These curves show a peak for the value of $\Omega_R = \sqrt{1-2\xi^2}$ characterising the resonance and therefore corresponding to the resonance pulsation

$\omega_R = \omega_0 \sqrt{1-2\xi^2}$ and to the resonance frequency $f_R = \dfrac{\omega_R}{2\pi}$. In this case the response is higher (or even much higher) than the static response.

It should be noted that resonance does not occur for $\omega = \omega_0$ but for $\omega = \omega_R$. Admitting that structural damping is weak in practice, we may consider that resonance occurs for $\omega = \omega_0$ and that amplification equals:

$$A(\Omega_R = 1) \approx \frac{1}{2\xi}$$

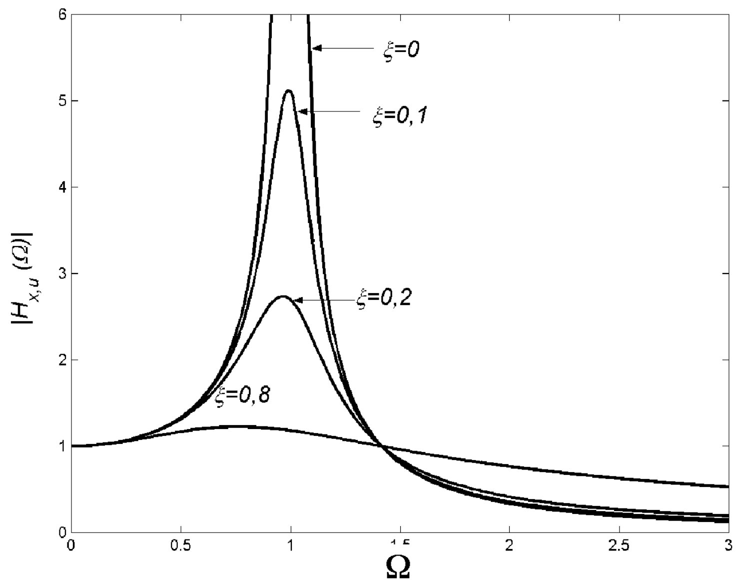


Figure 1.2: Resonance curves

Dimensioning of the structures based on dynamic loading cannot be made using only the maximum intensity of the load impact. Thus, for example, load $F(t) = F_0 \sin(\omega_1 t)$ can generate displacements or stresses very much lower than load $F(t) = (F_0/10) \sin(\omega_2 t)$ which however has an amplitude 10 times weaker, if this second load has a frequency much closer to the resonance frequency of the structure.

Resonance amplification being directly related to damping, it is therefore necessary to estimate this parameter correctly in order to obtain appropriate dynamic dimensioning. It should be noted that the simple oscillator study relies on the hypothesis of linear damping (viscous, with a damping force proportional to speed), which is one damping type among others. However, this is the assumption selected by most footbridge designers and engineers.

3.1.3 - Complex systems

The study of real structures, which are generally continuous and complex systems with an important number of degrees of freedom, may be considered as the study of a set of n simple oscillators, each one describing a characteristic vibration of the system. The approximation methods allowing such a conversion are detailed in Appendix 1. The new item with regard to the simple oscillator is the natural vibration mode defined by the pair constituted by the frequency and a vibratory shape (ω_i, ϕ_i) of the system. Computation of natural vibration modes is relatively intricate but designers nowadays have excellent software packages to obtain them, provided that they take, when modelling the system, all precautionary measures required for model analysis applications. Practical use of natural vibration modes is dealt with in chapter 3.

It should be emphasized that in some cases the problem can even be solved using a single simple oscillator. In any case, the main conclusions resulting from the simple oscillator study can be generalised to complex systems.

3.2 - Pedestrian loading

3.2.1 - Effects of pedestrian walking

Pedestrian loading, whether walking or running, has been studied rather thoroughly (see Appendix 2) and is translated as a point force exerted on the support, as a function of time and pedestrian position. Noting that x is the pedestrian position in relation to the footbridge centreline, the load of a pedestrian moving at constant speed v can therefore be represented as the product of a time component $F(t)$ by a space component $\delta(x - vt)$, δ being the Dirac operator, that is:

$$P(x,t) = F(t)\,\delta(x - vt)$$

Several parameters may also affect and modify this load (gait, physiological characteristics and apparel, ground roughness, etc.), but the experimental measurements performed show that it is periodic, characterised by a fundamental parameter: frequency, that is the number of steps per second. Table 1.1 provides the estimated frequency values.

Designation	Specific features	Frequency range (Hz)
Walking	Continuous contact with the ground	1.6 to 2.4
Running	Discontinuous contact	2 to 3.5

Table 1.1

Conventionally, for normal walking (unhampered), frequency may be described by a gaussian distribution with 2 Hz average and about 0.20 Hz standard deviation (from 0.175 to 0.22, depending on authors). Recent studies and conclusions drawn from recent testing have revealed even lower mean frequencies, around 1.8 Hz - 1.9 Hz.
The periodic function $F(t)$, may therefore be resolved into a Fourier series, that is a constant part increased by an infinite sum of harmonic forces. The sum of all unitary contributions of the terms of this sum returns the total effect of the periodic action.

$$F(t) = G_0 + G_1 \sin 2\pi f_m t + \sum_{i=2}^{n} G_i \sin(2\pi i f_m t - \varphi_i)$$

with G_0 : static force (pedestrian weight for the vertical component),
G_1 : first harmonic amplitude,
G_i : i-th harmonic amplitude,
f_m : walking frequency,
φ_i : phase angle of the i-th harmonic in relation to the first one,
n : number of harmonics taken into account.

The mean value of 700 N may be taken for G_0, weight of one pedestrian.
At mean frequency, around 2 Hz (f_m = 2 Hz) for vertical action, the coefficient values of the Fourier decomposition of $F(t)$ are the following (limited to the first three terms, that is n = 3, the coefficients of the higher of the terms being less than 0.1 G_0):

$G_1 = 0.4\, G_0$; $G_2 = G_3 \approx 0.1\, G_0$;

$\varphi_2 = \varphi_3 \approx \pi/2$.

By resolving the force into three components, that is, a «vertical» component and two horizontal components (one in the «longitudinal» direction of the displacement and one perpendicular to the transverse or lateral displacement), the following values of such components may be selected for dimensioning (in practice limited to the first harmonic):

Vertical component of one-pedestrian load:

$$F_v(t) = G_0 + 0{,}4\,G_0 \sin(2\pi f_m t)$$

Transverse horizontal component of one-pedestrian load:

$$F_{ht}(t) = 0{,}05\,G_0 \sin\left(2\pi\left(\frac{f_m}{2}\right)t\right)$$

Longitudinal horizontal component of one-pedestrian load:

$$F_{hl}(t) = 0{,}2\,G_0 \sin(2\pi f_m t)$$

It should be noted that, for one same walk, the transverse load frequency is equal to half the frequency of the vertical and longitudinal load. This is due to the fact that the load period is equal to the time between the two consecutive steps for vertical and longitudinal load since these two steps exert a force in the same direction whereas this duration corresponds to two straight and consecutive right footsteps or to two consecutive left footsteps in the case of transverse load since the left and right footsteps exert loads in opposite directions. As a result, the transverse load period is two times higher than the vertical and longitudinal load and therefore the frequency is two times lower.

3.2.2 - Effects of pedestrian running

Appendix 2 presents the effects of pedestrian running. This load case, which may be very dimensioning in nature, should not be systematically retained. Very often, the crossing duration of joggers on the footbridge is relatively short and does not leave much time for the resonance phenomenon to settle, in addition, this annoys the other pedestrians over a very short period. Moreover, this load case does not cover exceptional events such as a marathon race which must be studied separately. Accordingly, running effects shall not be considered in these guidelines.

3.2.3 - Random effects of several pedestrians and crowd

In practice, footbridges are submitted to the simultaneous actions of several persons and this makes the corresponding dynamic much more complicated. In fact, each pedestrian has its own characteristics (weight, frequency, speed) and, according to the number of persons present on the bridge, pedestrians will generate loads which are more or less synchronous with each other, on the one hand, and possibly with the footbridge, on the other. Added to these, there are the initial phase shifts between pedestrians due to the different moments when each individual enters the footbridge.

Moreover, the problem induced by intelligent human behaviour is such that, among others and facing a situation different to the one he expects, the pedestrian will modify his natural and normal gait in several ways; this behaviour can hardly be submitted to software processing.

It is therefore very difficult to fully simulate the actual action of the crowd. One can merely set out reasonable and simplifying hypotheses, based on pedestrian behaviour studies, and then assume that the crowd effect is obtained by multiplying the elementary effect of one pedestrian, possibly weighted by a minus factor. Various ideas exist as concern crowd effects and they antedate the Solférino and Millenium incidents. These concepts are presented in the following paragraphs together with a more comprehensive statistical study which will be used as a basis for the loadings recommended in these guidelines.

3.2.3.1 Random type pedestrian flow. Conventional model

For a large number of independent pedestrians (that is, without any particular synchronisation) which enter a bridge at a rate of arrival λ (expressed in persons/second) the average dynamic response at a given point of the footbridge submitted to this pedestrian flow is obtained by multiplying the effect of one single pedestrian by a factor $k = \sqrt{\lambda T}$, T being the time taken by a pedestrian to cross the footbridge (which can also be expressed by T = L/v where L represents the footbridge length and v the pedestrian speed). In fact, this product λT presents the number of N pedestrians present on the bridge at a given time. Practically, this means that n pedestrians present on a footbridge are equivalent to $\sqrt{N}$ all of them being synchronised. This result can be demonstrated by considering a crowd with the individuals all at the same frequency with a random phase distribution.
This result takes into consideration the phase shift between pedestrians, due to their different entrance time, but comprises a deficiency since it works on the assumption that all pedestrians are moving at the same frequency.

3.2.3.2 Experimental measurements on pedestrians flows

Several researchers have studied the forces and moments initiated by a group of persons, using measurements made on instrumented platforms where small pedestrian groups move.
Ebrahimpour & al. (Ref. [24]) propose a sparse crowd loading model on the first term of a Fourier representation the coefficient α_1 of which depends on the number N_p of persons present on the platform (for a 2 Hz walking frequency):
$\alpha_1 = 0.34 - 0.09\, log(n_p)$ for $N_p < 10$
$\alpha_1 = 0.25$ for $N_p > 10$
Unfortunately, this model does not cover the cumulative random effects.

3.2.3.3 Comprehensive simulation model of pedestrian flows

Until recently dynamic dimensioning of footbridges was mainly based on the theoretical model loading case with one single pedestrian completed by rather crude requirements concerning footbridge stiffness and natural frequency floor values. Obviously, such requirements are very insufficient and in particular they do not cover the main problems raised by the use of footbridges in urban areas which are subject to the action of more or less dense pedestrian groups and crowds. Even the above-addressed $\sqrt{N}$ model has some deficiencies.
It rapidly appears that knowledge of crowd behaviour is limited and this makes the availability of practical dimensioning means all the more urgent. It is better to suggest simple elements to be improved on subsequently, rather than remaining in the current knowledge void.
Therefore several crowd load cases have been developed using probability calculations and statistical processing to deepen the random crowd issue. The model finally selected consists in handling pedestrians' moves at random frequencies and phases, on a footbridge presenting different modes and in assessing each time the equivalent number of pedestrians which - when evenly distributed on the footbridge, or in phase and at the natural frequency of the footbridge will produce the same effect as random pedestrians.

Several digital tests were performed to take into consideration the statistical effect. For each test including N pedestrians and for each pedestrian, a random phase φ and a normally distributed random frequency $f = \frac{\omega}{2\pi}$ centred around the natural frequency of the footbridge and with 0.175 Hz standard deviation, are selected; the maximum acceleration over a sufficiently long period (in this case the time required for a pedestrian to span the footbridge twice at a 1.5m/s speed) is noted and the equivalent number of pedestrians which would be perfectly synchronised is calculated. The method used is explained in figure 1.3.

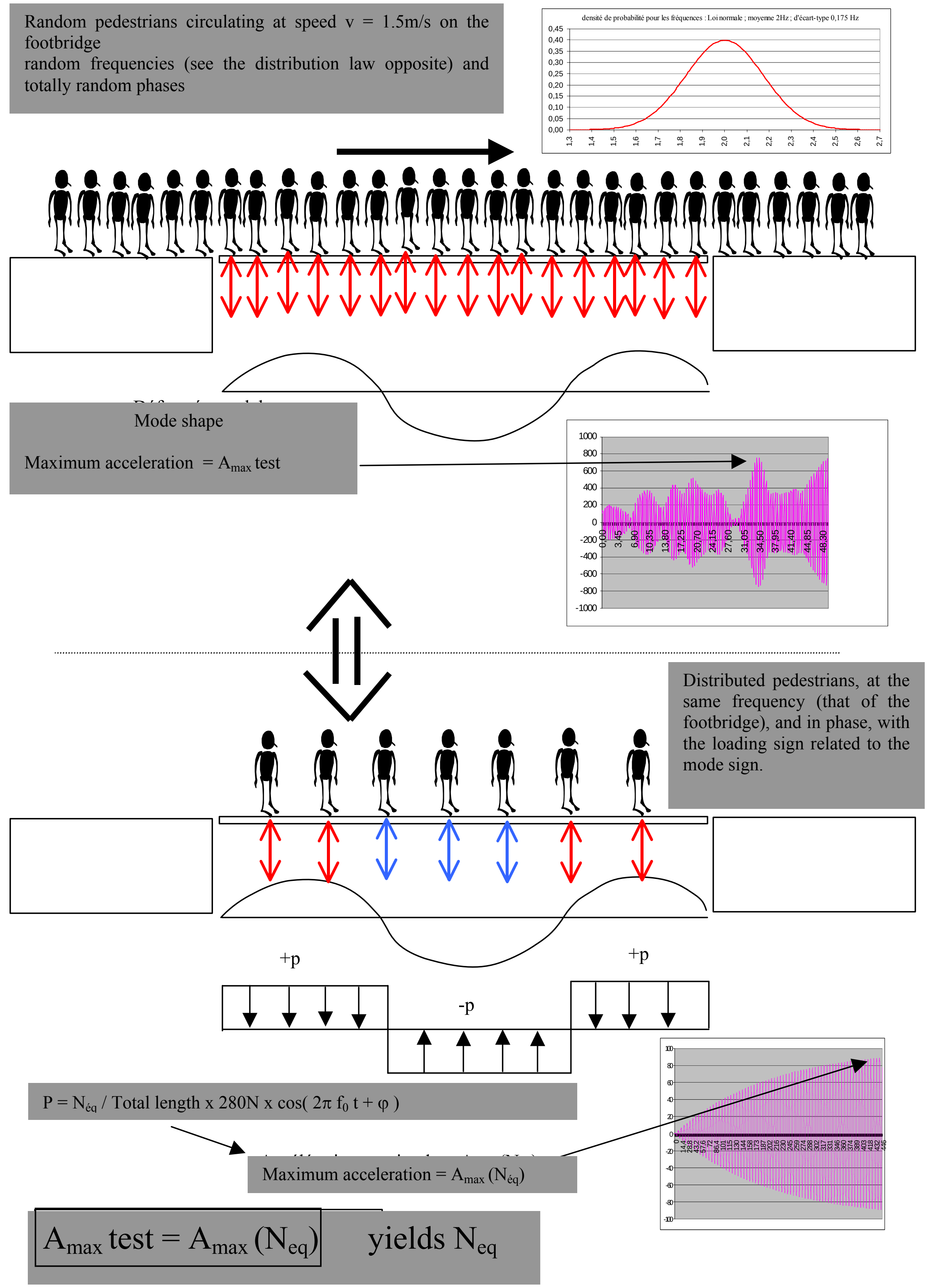


Figure 1.3: Calculation methodology for the equivalent number of pedestrians N_{eq}

These tests are repeated 500 times with a fixed number of pedestrians, fixed damping and a fixed number of mode antinodes; then the characteristic value, such as 95% of the samples give a value lower than this characteristic value (95% characteristic value, 95 percentile or 95% fractile). This concept is explained in figure 1.4:

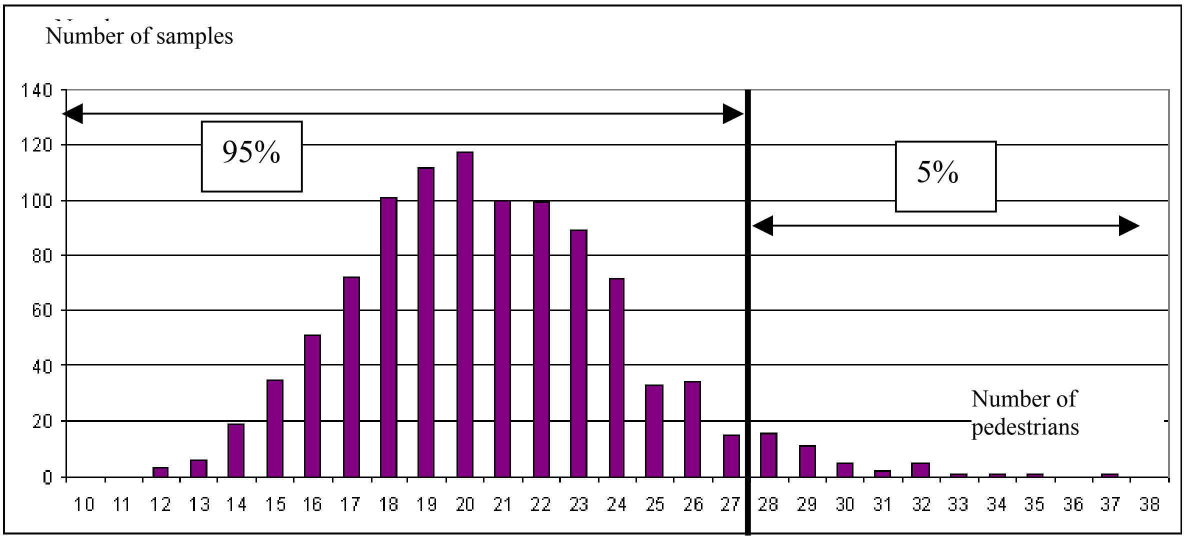


Figure 1.4: 95% fractile concept

By varying damping, the number of pedestrians, the number of mode antinodes, it is possible to infer a law for the equivalent number of pedestrians, this law is the closest to the performed test results.

The following two laws are retained:

Sparse or dense crowd: random phases and frequencies with Gaussian law distribution: $N_{eq} = 10{,}8\sqrt{N\xi}$ where N is the number of pedestrians present on the footbridge (density × surface area) and ξ the critical damping ratio.

Very dense crowd: random phases and all pedestrians at the same frequency: $N_{eq} = 1{,}85\sqrt{N}$.

This model is considerably simplified in the calculations. We just have to distribute the N_{eq} pedestrians on the bridge, to apply to these pedestrians a force the amplitude sign is the same as the mode shape sign and to consider this force as the natural frequency of the structure and to calculate the maximum acceleration obtained at the corresponding resonance. Chapter 3 explains how this loading is taken into consideration.

3.2.4 - Lock-in of a pedestrian crowd

Lock-in expresses the phenomenon by which a pedestrian crowd, with frequencies randomly distributed around an average value and with random phase shifts, will gradually coordinate at common frequency (that of the footbridge) and enters in phase with the footbridge motion.
So far, known cases of crowd lock-in have been limited to transverse footbridge vibrations. The most recent two cases, now famous, are the Solférino footbridge and the Millenium footbridge which were submitted to thorough in-situ tests. Once again, these tests confirm that the phenomenon is clearly explained by the pedestrian response as he modifies his walking pace when he perceives the transverse motion of the footbridge and it begins to disturb him. To compensate his incipient unbalance, he instinctively follows the footbridge motion frequency. Thus, he directly provokes the resonance phenomenon and since all pedestrians

undergo it, the problem is further amplified and theoretically the whole crowd may become synchronised. Fortunately, on the one hand, actual synchronisation is much weaker and, on the other, when the footbridge movement is such that the pedestrians can no longer put their best foot forward, they have to stop walking and the phenomenon can no longer evolve.

3.2.4.1 Pedestrian flows measured on a real footbridge structure

Using a large footbridge with a 5.25m x 134m main span which can be submitted to a very dense crowd (up to about 2 persons/m²), Fujino & al. (Ref. [30]) observed that application of the above factor gave an under-estimation of about $\sqrt{N}$ 1 to 10 times of the actually observed lateral vibration amplitude. They formed the hypothesis of synchronisation of a crowd walking in synchrony with the transverse mode frequency of their footbridge to explain the phenomenon and were thus able to prove, in this case, the measurement magnitude obtained. This is the phenomenon we call "lock-in", and a detailed presentation is provided below.
For this structure, by retaining only the first term of the Fourier decomposition for the pedestrian-induced load, these authors propose a $0.2N$ multiplication factor to represent any loading which would equate that of a crowd of N persons, allowing them to retrieve the magnitude of the effectively measured displacements (0.01 m).

3.2.4.2 Theory formulated for the Millenium footbridge

Arup's team issued a very detailed article on the results obtained following this study and tests performed on the Millenium footbridge. (Ref. [38]). Only the main conclusions of this study are mentioned here.
The model proposed for the Millenium footbridge study is as follows: the force exerted by a pedestrian (in N) is assumed to be related to footbridge velocity.
$F_{1 pedestrian} = KV(x,t)$ where K is a proportionality factor (in Ns/m) and V the footbridge velocity at the point x in question and at time t.
Seen this way, pedestrian load may be understood as a negative damping. Assuming a viscous damping of the footbridge, the negative damping force induced by a pedestrian is directly deducted from it. The consequence of lock-in is an increase of this negative damping force, induced by the participation of a higher number of pedestrians. This is how the convenient notion of critical number appears: this is the number of pedestrians beyond which their cumulative negative damping force becomes higher than the inherent damping of the footbridge; the situation would then be similar to that of an unstable oscillator: a small disturbance may generate indefinitely-amplifying movements…
For the particular case of a sinusoidal horizontal vibration mode (the maximum amplitude of this mode being normalized to 1, f_1 representing the first transverse natural frequency and m_1 the generalised mass in this mode, considering the maximum unit displacement) and assuming an uneven distribution of the pedestrians, the critical number can then be written as:

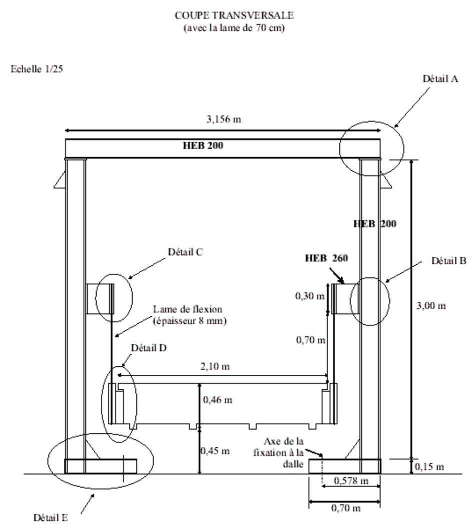
$$N = \frac{8\pi\xi m_1 f_1}{K}$$

K is the proportionality factor, with a value of 300 Ns/m in the case of the Millenium footbridge.
We can then note that a low damping, a low mass, or a low frequency is translated by a small critical number and therefore a higher lock-in risk. Consequently, to increase the critical number it will be necessary to act on these three parameters.
It will be noted that the value of factor K cannot *a priori* be generalised to any structure and therefore its use increases the criterion application uncertainty.

3.2.4.3 Laboratory tests on platform

To quantify the horizontal load of a pedestrian and the pedestrian lock-in effects under lateral motion, some tests were carried out on a reduced footbridge model by recreating, using a dimensional analysis, the conditions prevailing on a relatively simple design of footbridge (one single horizontal mode).

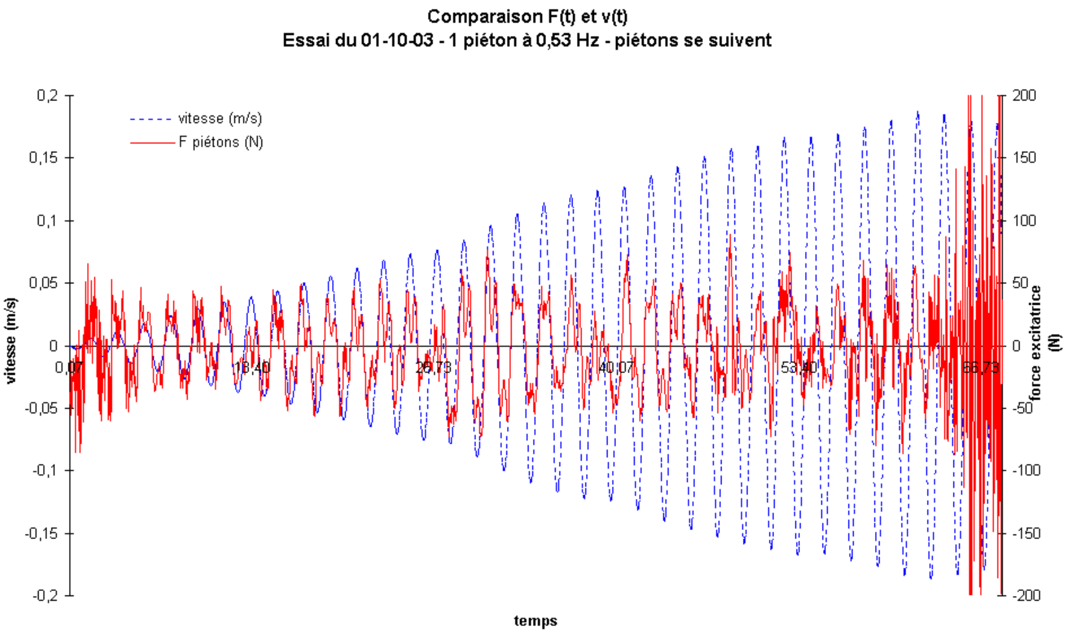
The principle consists in placing a 7-metre long and 2-metre wide slab on 4 flexible blades moving laterally and installing access and exit ramps as well as a loop to maintain walking continuity (Photos 1.1 and 1.2). To maintain this continuity, a large number of pedestrians is of course needed on the loop; this number being clearly higher than the number of pedestrians present on the footbridge at a given time.



CROSS-SECTION VIEW (with blade 70 cm)
Scale 1/25
Detail A
Detail B
Detail C
Flexible blade (thickness 8 mm)
Detail D
Attachment axis to the slab
Detail E
HEB

Photographs 1.1 and 1.2: Description of the model

By recreating the instantaneous force from the displacements measured (previously filtered to attenuate the effect of high frequencies) ($F(t) = m\ddot{x}(t) + c\dot{x}(t) + kx(t)$). Figure 1.5 shows that, in a first step, for an individual pedestrian the amplitude of the pedestrian force remains constant, around 50 N, and in any case, lower than 100N, whatever the speed amplitude. In a second step, we observe that the force amplitude increases up to 150 N, but these last oscillations should not be considered as they represent the end of the test.



Comparison of F(t) and v(t)
Test of 01-10-03 – 1 pedestrian at 0.53 Hz – pedestrians following each other
Speed (m/s)
F pedestrians (N)
Time
Exciting force (N)

Figure 1.5: Force and speed at forced resonant rate

We have a peak value which does not exceed 100 N and is rather around 50N on average, with the first harmonic of this signal being around 35 N.

The following graphs (figures 1.6 and 1.7) represent, on the same figure, the accelerations in time (pink curve and scale on the RH side expressed in m/s², variation from 0.1 to 0.75 m/s²) and the «efficient» force (instantaneous force multiplied by the speed sign, averaged on a period, which is therefore positive when energy is injected into the system and negative in the opposite case) for a group of pedestrians (blue curve, scale on the LH side expressed in N).

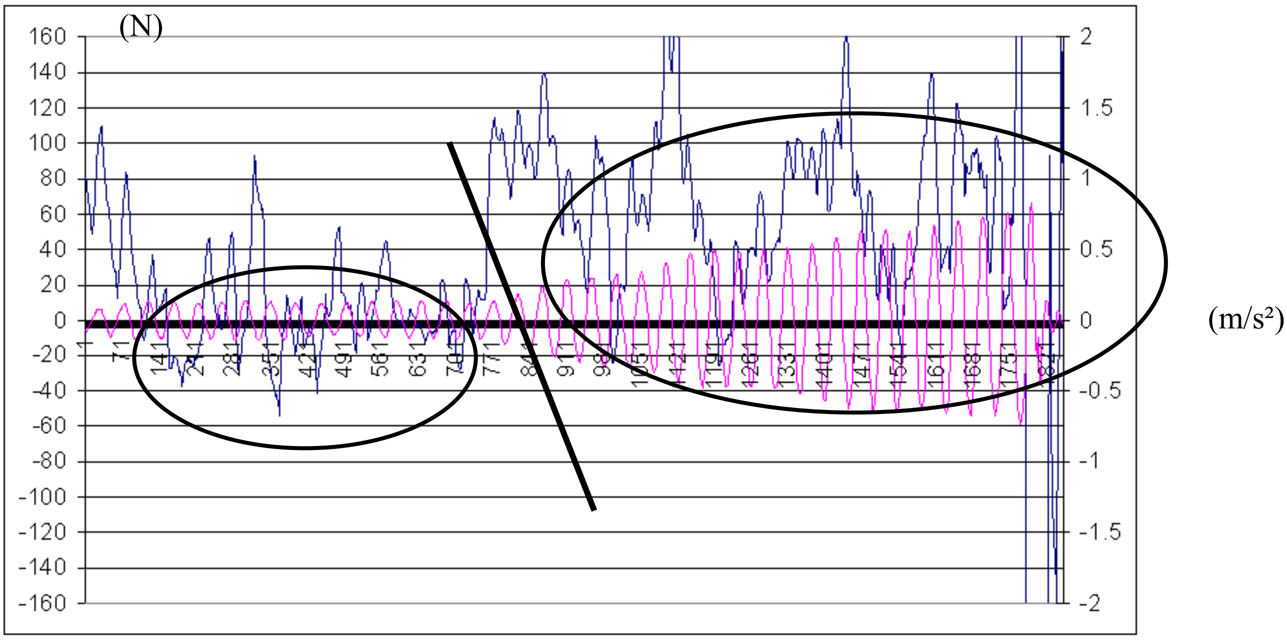


Figure 1.6: Acceleration (m/s²) and efficient force (N) with 6 random pedestrians on the footbridge

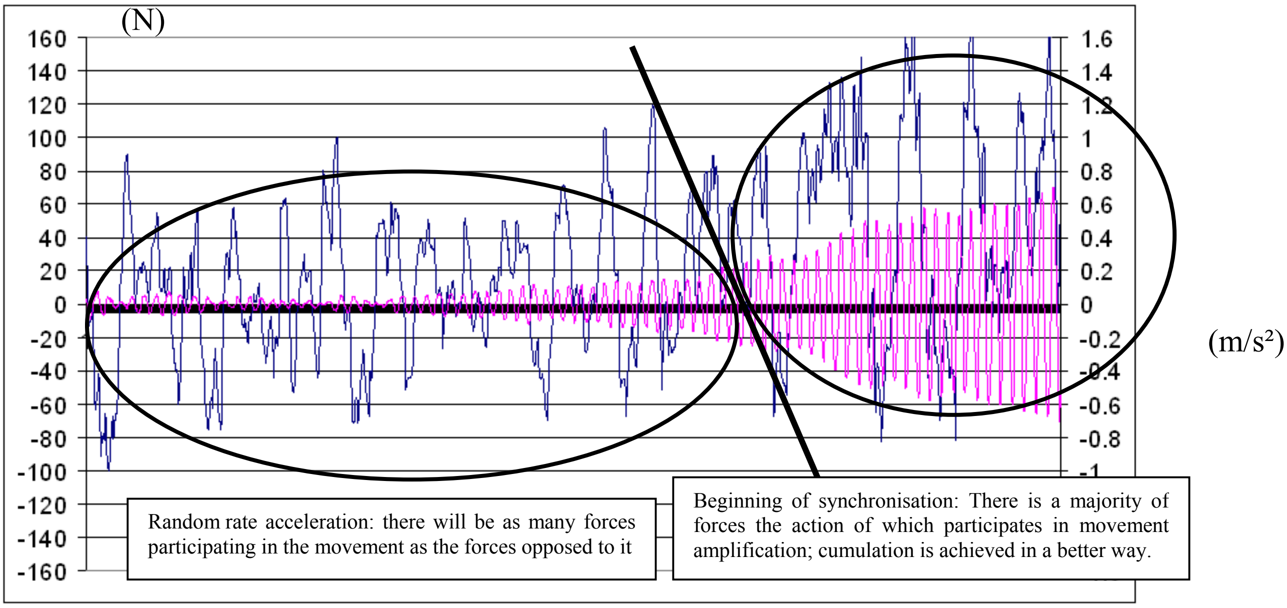


Figure 1.7: Acceleration (m/s²) and efficient force (N) with 10 random pedestrians on the footbridge

We observe that, from a given value, the force exerted by the pedestrians is clearly more efficient and there is some incipient synchronisation. This threshold is around 0.15 m/s² (straight line between the random rate zone and the incipient synchronisation zone). However, there is only some little synchronisation (maximum value of 100-150 N i.e. 0.2 to 0.3 times the effect of 10 pedestrians), but this is quite sufficient to generate very uncomfortable vibrations (>0.6m/s²).

3.2.4.4 Experience gained from the Solferino footbridge test results

Several test campaigns were carried out over several years following the closing of the Solferino footbridge to traffic, from the beginning these tests were intended to identify the issues and develop corrective measures; then they were needed to check the efficiency of the adopted measures and, finally, to draw lessons useful for the scientific and technical community.

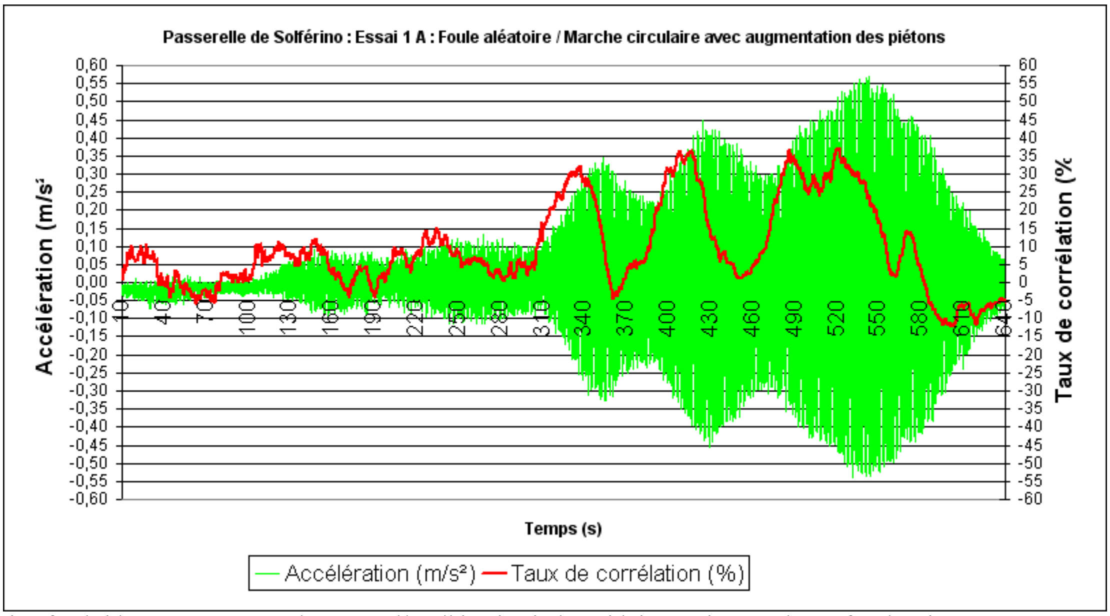
The main conclusions to be drawn from the Solferino footbridge tests are the following:

- The lock-in phenomenon effectively occurred for the first mode of lateral swinging for which the double of the frequency is located within the range of normal walking frequency of pedestrians.
- On the other hand, it does not seem to occur for modes of torsion that simultaneously present vertical and horizontal movements, even when the test crowd was made to walk at a frequency that had given rise to resonance. The strong vertical movements disturb and upset the pedestrians' walk and do not seem to favour maintaining it at the resonance frequency selected for the tests. High horizontal acceleration levels are then noted and it seems their effects have been masked by the vertical acceleration.
- The concept of a critical number of pedestrians is entirely relative: it is certain that below a certain threshold lock-in cannot occur, however, on the other hand, beyond a threshold that has been proven various specific conditions can prevent it from occurring.
- Lock-in appears to initiate and develop more easily from an initial pedestrian walking frequency for which half the value is lower than the horizontal swinging risk natural frequency of the structure. In the inverse case, that is, when the walking crowd has a faster initial pace several tests have effectively shown that it did not occur. This would

be worth studying in depth but it is already possible to explain that the pedestrian walking fairly rapidly feels the effects of horizontal acceleration not only differently, which is certain, but also less noticeably, and this remains to be confirmed.

- Clearly lock-in occurs beyond a particular threshold. This threshold may be explained in terms of sufficient number of pedestrians on the footbridge (conclusion adopted by Arup's team), but it could just as well be explained by an acceleration value felt by the pedestrian, which is more practical for defining a verification criterion.

The following graphs (figures 1.8 to 1.13) present a summary of the tests carried out on the Solferino footbridge. The evolution of acceleration over time is shown (in green), and in parallel the correlation or synchronisation rate, ratio between the equivalent number of pedestrians and the number of pedestrians present on the footbridge. The equivalent number of pedestrians can be deduced from the instantaneous modal force. It is the number of pedestrians who, regularly distributed on the structure, and both in phase and at the same frequency apparently inject an identical amount of energy per period into the system.



Solferino footbridge: Test 1 A: Random crowd/Walking in circles with increasing numbers of pedestrians
Acceleration (m/s)
Correlation rate (%)
Time (s)
…… Acceleration (m/s²) ------ Correlation rate (%)

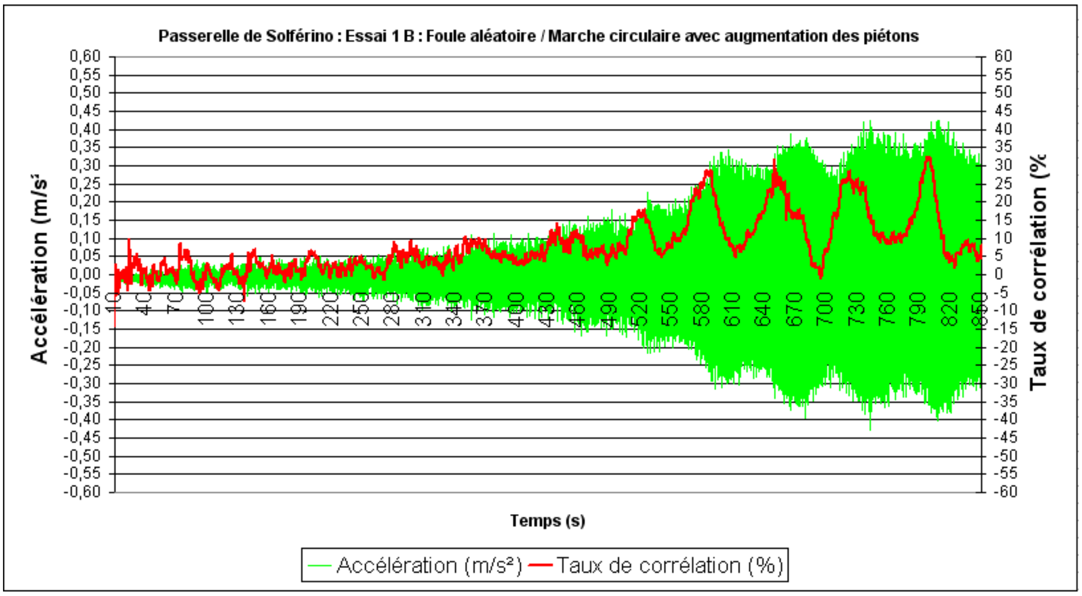
Figure 1.8: 1A Solferino footbridge random test: a crowd is made to circulate endlessly on the footbridge with the number of pedestrians being progressively increased (69 – 138 – 207).

In the test shown in Figure 1.8, it can be seen that below 0.12m/s², behaviour is completely random, and from 0.15m/s², it becomes partly synchronised, with synchronisation reaching 30-35% when the acceleration amplitudes are already high (0.45m/s²). The concept of rate change critical threshold (shift from a random rate to a partly synchronised rate) becomes perceptible.

The various "loops" correspond to the fact that the pedestrians are not regularly distributed on the footbridge, they are concentrated in groups. For this reason, it is clear when the largest group of pedestrians is near the centre of the footbridge (sag summit), or rather at the ends of the footbridge (sag trough).

It can also be seen that the three acceleration rise sags, that occur at increasing levels of acceleration (0.3m/s² then 0.4m/s² and finally 0.5m/s²), occur with the same equivalent number of pedestrians each time. This clearly shows the fact that there is an acceleration rise, halted twice when the group of pedestrians reaches the end of the footbridge.

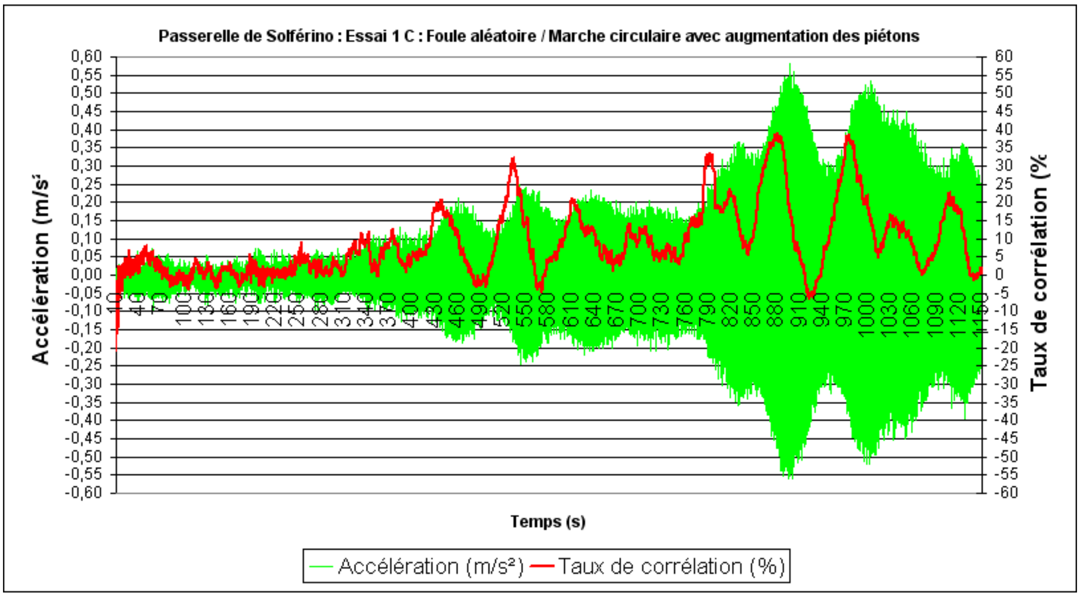
In the following test, shown in Figure 1.9, the number of pedestrians has been increased more progressively.



Solferino footbridge: Time 1 B: Random crowd/Walking in circles with increasing numbers of pedestrians
Acceleration (m/s)
Correlation rate (%)
Time (s)
…… Acceleration (m/s²) ------ Correlation rate (%)

Figure 1.9: 1B Solferino footbridge random test: a crowd is made to circulate endlessly on the footbridge with the number of pedestrians being more progressively increased (115 138 161 92 – 184 – 202).

This time, the rate change threshold seems to be situated around 0.15 – 0.20 m/s². Maximum synchronisation rate does not exceed 30%.

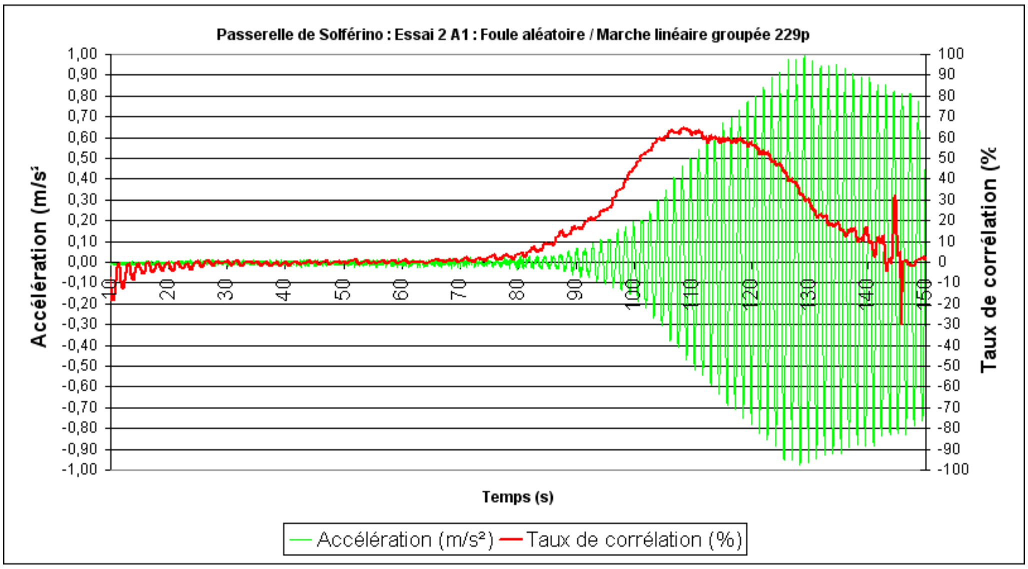


Solferino footbridge: Test 1 C: Random crowd/Walking in circles with increasing numbers of pedestrians
Acceleration rate (m/s)

Correlation rate (%)
Time (s)
…… Acceleration (m/s²) ------ Correlation rate (%)

Figure 1.10: 1C Solferino footbridge random test

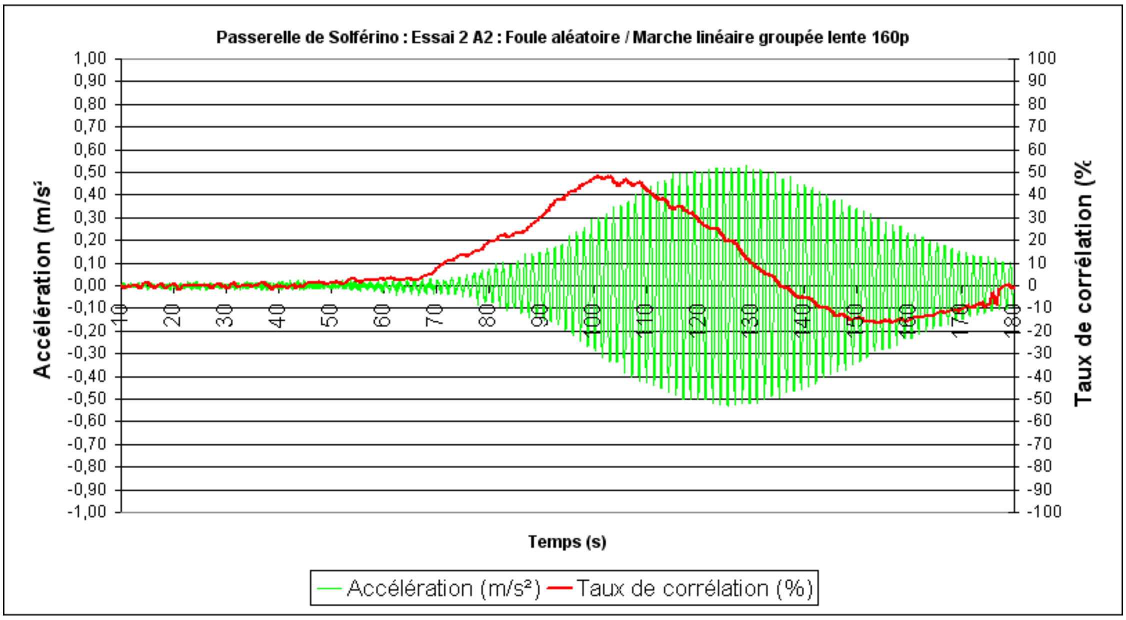
Test 1C, shown in Figure 1.10, leads us to the same conclusions: change in threshold between 0.10 and 0.15m/s², then more obvious synchronisation reaching up to 35%-40%.



Solferino footbridge: Test 2 A1: Random crowd/Walking grouped together in a straight line 229p
Acceleration (m/s)
Correlation rate (%)
Time (s)
…… Acceleration (m/s²) ------ Correlation rate (%)

Figure 1.11: 2A1 Solferino footbridge random test

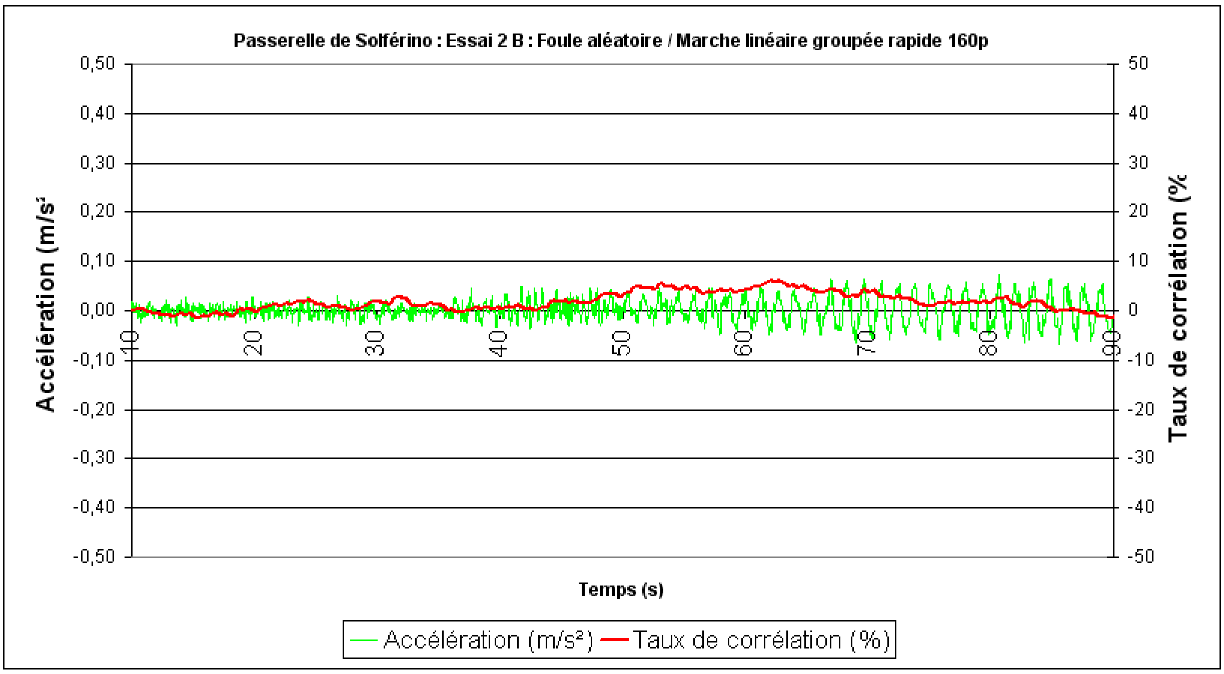
In the test shown in Figure 1.11, the pedestrians are more grouped together and walk from one edge of the footbridge to the other. The rise and subsequent fall in equivalent number of pedestrians better expresses the movement of pedestrians, and their crossing from an area without displacement (near the edges) and another with a lot of displacement (around mid-span). Synchronisation rate rises to about 60%. This is higher than previously, however, it should be pointed out, on the one hand, that the level of vibration is higher (0.9m/s² instead of 0.5m/s²) and, on the other, that the crowd is fairly compact and this favours synchronisation phenomenon among the pedestrians.



Solferino footbridge: Test 2 A2: Random crowd/Walking grouped together in a straight line 160p
Acceleration (m/s)
Correlation rate (%)
Time (s)
…… Acceleration (m/s²) ------ Correlation rate (%)

Figure 1.12: 2A2 Solferino footbridge random test

In the test shown in Figure 1.12, there are only 160 people still walking slowly from one end of the footbridge to the other. This time, synchronisation rate reaches 50%. Accelerations are comparable with those obtained in tests 1A, 1B, and 1C.



Solferino footbridge: Test 2 B: Random crowd/Rapid walking in straight line 160p
Acceleration (m/s)
Correlation rate (%)
Time (s)
…… Acceleration (m/s²) ------ Correlation rate (%)

Figure 1.13: 2B Solferino footbridge random test

This last test (figure 1.13) is identical to the previous one except that, this time, the pedestrians were walking fast. Pedestrian synchronisation phenomenon was not observed, although there was a compact crowd of 160 people. This clearly shows that where pedestrian walking frequency is too far from the natural frequency, synchronisation does not occur.

These tests show that there is apparently a rate change threshold in relation to random rate at around 0.10-0.15 m/s². Once this threshold has been exceeded accelerations rise considerably but remain limited. Synchronisation rates in the order of 30 to 50% are reached when the test is stopped. This value can rise to 60%, or even higher, when the crowd is compact.

For dimensioning purposes, the value 0.10 m/s² shall be noted. Below this threshold, behaviour of pedestrians may be qualified as random. It will then be possible to use the equivalent random pedestrian loadings mentioned above and this will lead to synchronisation rates to the order of 5 to 10%. Synchronisation rate can rise to more than 60% once this threshold has been exceeded. Thus, acceleration goes from 0.10 to over 0.60 m/s² relatively suddenly. Acceleration thus systematically becomes uncomfortable. Consequently, the 0.10 m/s² rate change threshold becomes a threshold not to be exceeded.

3.2.4.5 Conclusions on lock-in

The various studies put forward conclusions that appear to differ but that actually concur on several points.
As soon as the amplitude of the movements becomes perceptible, crowd behaviour is no longer random and a type of synchronisation develops. Several models are available (force as a function of speed, high crowd synchronisation rate) but they all lead to accelerations rather in excess of the generally accepted comfort thresholds.
Passage from a random rate on a fixed support to a synchronised rate on a mobile support occurs when a particular threshold is exceeded, characterised by critical acceleration or a critical number of pedestrians. It should be noted that the concept of a critical number of pedestrians and the concept of critical acceleration may be linked. Critical acceleration may be interpreted as the acceleration produced by the critical number of pedestrians, still random though after this they are no longer so.
Although it is true that the main principles and the variations in behaviour observed on the two footbridges concur, modelling and quantitative differences nevertheless lead to rather different damper dimensioning.
The concept of critical acceleration seems more relevant that that of a critical number of pedestrians. Acceleration actually corresponds to what the pedestrians feel whereas a critical number of pedestrians depends on the way in which the said pedestrians are organised and positioned on the footbridge. It is therefore this critical acceleration threshold that will be discussed in these guidelines, the way in which the pedestrians are organised depends on the level of traffic on the footbridge (see below).

3.3 - Parameters that affect dimensioning: frequency, comfort threshold, comfort criterion, etc.

The problems encountered on recent footbridges echo the well-known phenomenon of resonance that ensues from the matching of the exciting frequency of pedestrian footsteps and the natural frequency of a footbridge mode. Owing to the fact that they are amplified, noticeable movements of the footbridge ensue and their consequence is a feeling of discomfort for the pedestrians that upsets their progress.
Thus, it is necessary to review the structural parameters, vital to the resonance phenomenon, represented by natural vibration modes (natural modes and natural frequencies) and the values of the structural critical damping ratio associated with each mode. In reality, even a footbridge of simple design will have an infinity of natural vibration modes, frequencies and critical damping ratios associated to it (see Appendix 1). However, in most cases, it is sufficient to study a few first modes.
Along with this, it is necessary to consider pedestrian walking frequencies, since they differ from one individual to another, together with walking conditions and numerous other factors. So, it is necessary to bear in mind a range of frequencies rather than a single one.
The first simple method for preventing the risk of resonance could consist in avoiding having one or several footbridge natural frequencies within the range of pedestrian walking frequency. This leads to the concept of a range of risk frequencies to be avoided.

3.3.1 - Risk frequencies noted in the literature and in current regulations

Compilation of the frequency range values given in various articles and regulations has given rise to the following table, drawn up for vertical vibrations:

Eurocode 2 (Ref. [4])	1.6 Hz and 2.4 Hz and, where specified, between 2.5 Hz and 5 Hz.
Eurocode 5 (Ref. [5])	Between 0 and 5 Hz
Appendix 2 of Eurocode 0	<5 Hz
BS 5400 (Ref. [6])	<5 Hz
Regulations in Japan (Ref. [30])	1.5 Hz – 2.3 Hz
ISO/DIS standard 10137 (Ref. [28])	1.7 Hz – 2.3 Hz
CEB 209 Bulletin	1.65 – 2.35 Hz
Bachmann (Ref. [59])	1.6 – 2.4 Hz

Table 1 .2

As concerns lateral vibrations, the ranges described above are to be divided by two owing to the particular nature of walking: right and left foot are equivalent in their vertical action, but are opposed in their horizontal action and this means transverse efforts apply at a frequency that is half that of the footsteps.
However, on the Millenium footbridge it was noticed that the lock-in phenomenon appeared even for a horizontal mode with a frequency considerably beneath that of the lower limit generally accepted so far for normal walking frequency. Thus, for horizontal vibration modes, it seems advisable to further lower the lower boundary of the risk frequency range.
Although risk frequency ranges are fairly well known and clearly defined, in construction practice, it is not easy to avoid them without resorting to impractical rigidity or mass values. Where it is impossible to avoid resonance, it is necessary to try to limit its adverse effects by

acting on the remaining parameter: structural damping; obviously, it will be necessary to have available criteria making it possible to determine the acceptable limits of the resonance.

3.3.2 - Comfort thresholds

Before going any further, the concept of comfort should be specified. Clearly, this concept is highly subjective. In particular:

- from one individual to the next, the same vibrations will not be perceived in the same way.
- for a particular individual, several thresholds may be defined. The first is a vibration perception threshold. This is followed by a second that can be related to various degrees of disturbance or discomfort (tolerable over a short period, disturbing, unacceptable). Finally, a third threshold may be determined in relation to the consequences the vibrations may entail: loss of balance, or even health problems.
- furthermore, depending on whether he is standing, seated, moving or stationary, a particular individual may react differently to the vibrations.
- it is also well known that there is a difference between the vibrations of the structure and the vibrations actually perceived by the pedestrian. For instance, the duration for which he is exposed to the vibrations affects what the pedestrian feels. However, knowledge in this field remains imprecise and insufficient.

Hence, though highly desirable, it is clear that in order to be used for footbridge dimensioning, determining thresholds in relation to the comfort perceived by the pedestrian is a particularly difficult task. Indeed, where a ceiling of $1 m/s^2$ for vertical acceleration has been given by Matsumoto and al, (Ref. [16]), others, such as Wheeler, on the one hand, and Tilly et al, on the other, contradict each other with values that are either lower or higher (Ref. [18] and Ref. [20]). Furthermore, so far, there have only been a few suggestions relating to transverse movements.
Hence, where projects are concerned, it is necessary to consider the recommended values as orders of magnitude and it is more convenient to rely on parameters that are easy to calculate or measure; in this way, the thresholds perceived by the pedestrians are assimilated with the measurable quantities generated by the footbridges and it is generally the peak acceleration value, called the critical acceleration a_{crit} of the structure, that is retained; the comfort criterion to be respected is represented by this value that should not be exceeded. Obviously, it is necessary to define appropriate load cases to apply to the structure in order to determine this acceleration. This verification is considered to be a service limit state.

3.3.3 - Acceleration comfort criteria noted in the literature and regulations

The literature and various regulations put forward various values for the critical acceleration, noted a_{crit}. The values are provided for vertical acceleration and are shown in Figure 1.14 below:

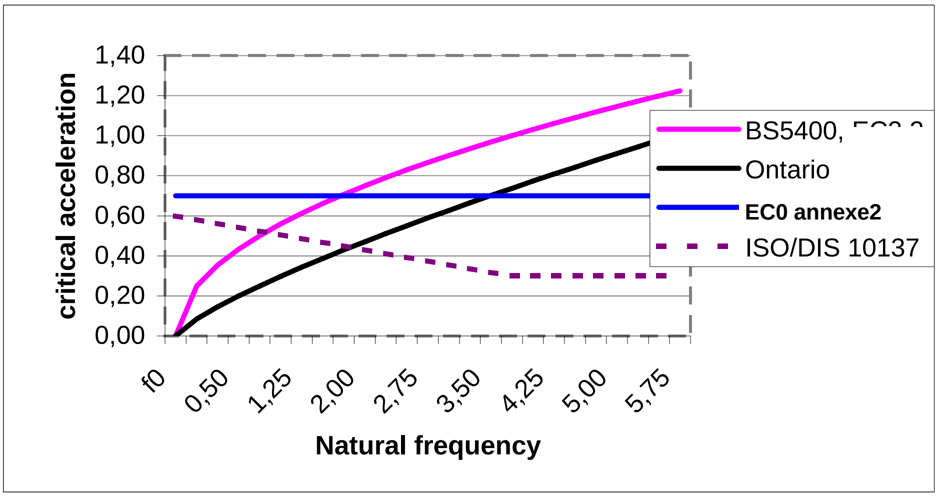


Figure 1.14: Vertical critical accelerations (in m/s²) as a function of the natural frequency for various regulations: some depend on the frequency of the structure, others do not.

For vertical vibrations with a frequency of around 2 Hz, standard walking frequency, there is apparently a consensus for a range of 0.5 to 0.8 m/s². It should be remembered that these values are mainly associated with the theoretical load of a single pedestrian.
For lateral vibrations with a frequency of around 1 Hz, Eurocode 0 Appendix 2 proposes a horizontal critical acceleration of 0.2 m/s² under normal use and 0.4 m/s² for exceptional conditions (i.e. a crowd). Unfortunately, the text does not provide crowd loadings.
It is also useful to remember that since accelerations, speeds and displacements are related, an acceleration threshold may be translated as a displacement threshold (which makes better sense for a designer) or even a speed threshold.

- $Acceleration = (frequency\ of\ 2Hz)^2\ movement$
- $Acceleration = (frequency\ of\ 2Hz)\ speed$

For instance,
for a frequency of 2 Hz:
acceleration of 0.5m/s² corresponds to a displacement of 3.2mm, a speed of 0.04m/s
acceleration of 1m/s² corresponds to a displacement of 6.3 mm, a speed of 0.08 m/s
but for a frequency of 1 Hz:
acceleration of 0.5m/s² corresponds to a displacement of 12.7mm, a speed of 0.08m/s
acceleration of 1m/s² corresponds to a displacement of 25,3mm, a speed of 0.16 m/s

3.4 - Improvement of dynamic behaviour

What is to be done when footbridge acceleration does not respect comfort criteria?

It is necessary to distinguish between a footbridge at the design stage and an existing one.

In the case of a footbridge at the design stage, it is logical to try to modify its natural frequency vibrations. If it is not possible to modify them so that they are outside the resonance risk ranges in relation to excitation by the pedestrians, then attempts should be made to increase structural damping.

With an existing footbridge, it is also possible to try to modify its natural frequency vibrations. However, experience shows that it is generally cheaper to increase damping.

3.4.1 - Modification of vibration natural frequencies

A vibration natural frequency is always proportional to the square root of the stiffness and inversely proportional to the square root of the mass. The general aim is to try to increase vibration frequency. Therefore the stiffness of the structure needs to be increased. However, practice indicates that an increase in stiffness is frequently accompanied by an increase in mass, which produces an inverse result, this is a difficult problem to solve.

3.4.2 - Increasing structural damping

3.4.2.1 Natural structural damping of the structures

The critical damping ratio is not an inherent fact of a material. Most experimental results suggest that dissipation forces are to all practical intents and purposes independent of frequency but rather depend on movement amplitude. The critical damping ratio also increases when vibration amplitude increases. It also depends on construction details that may dissipate energy to a greater or lesser extent (for instance, where steel is concerned, the difference between bolting and welding).

It should be noted that although the mass and rigidity of the various structure elements may be modelled with a reasonable degree of accuracy, damping properties are far more difficult to characterise. Studies generally use critical damping coefficients ranging between 0.1% and 2.0% and it is best not to overestimate structural damping in order to avoid under-dimensioning.

CEB information bulletin No. 209 (Ref. [7]), an important summary document dealing with the general problem of structure vibrations provides the following values for use in projects:

Type of deck	Critical damping ratio	
	Minimum value	Average value
Reinforced concrete	0.8%	1.3%
Prestressed concrete	0.5%	1.0%
Metal	0.2%	0.4%
Mixed	0.3%	0.6%
Timber	1.5%	3.0%

Table 1.3

As concerns timber, Eurocode 5 recommends values of 1% or 1.5% depending on the presence, or otherwise, of mechanical joints.

Where vibration amplitude is high, as with earthquakes, critical damping ratios are considerably higher and need to be SLS checked. For instance, the AFPS 92 Guide on seismic protection of bridges (art. 4.2.3) states the following:

Material	Critical damping ratio
Welded steel	2%
Bolted steel	4%
Prestressed concrete	2%
Non-reinforced concrete	3%
Reinforced concrete	5%
Reinforced elastomer	7%

Table 1.4

Finally, it should be pointed out that an estimate of actual structural damping can only be achieved through measurements made on the finished structure. This said, increased damping may be obtained from the design stage, for instance through use of a wire-mesh structure, or in the case of tension tie footbridges («stress ribbon»), through insertion of elastomer plates distributed between the prefabricated concrete slabs making up the deck.

3.4.2.2 Damper implementation

The use of dampers is another effective solution for reducing vibrations. Appendix 3 describes the different types of dampers that can be used and describes the operating and dimensioning principle of a selection of dampers. The table of examples below shows this is a tried and tested solution to the problem:

Country	Name	Mass kg	Total effective mass %	Damper critical damping ratio	Critical damping ratio of the structure without dampers	Critical damping ratio of the structure with dampers	Structure frequency Hz
France	Passerelle du Stade de France (Football stadium footbridge) (Saint-Denis)	2400 per span	1.6	0.075	0.2% to 0.3%	4.3% to 5.3%	1.95 (vertical)
France	Solferino footbridge (Paris)	15000	4.7		0.4%	3.5%	0.812 (horizontal)
Ditto	Ditto	10000	2.6		0.5%	3%	1.94 (vertical)
Ditto	Ditto	7600	2.6		0.5%	2%	2.22 (vertical)
England	Millenium footbridge (London)	2500			0.6% to 0.8%	2%	0.49 (horizontal)
Ditto	Ditto	1000 to 2500			0.6% to 0.8%		0.5 (vertical)
Japan			1		0.2%	2.2%	1.8 (vertical)
USA	Las Vegas (Bellagio-Bally)				0.5%	8%	
South Korea	Seonyu footbridge				0.6%	3.6%	0.75 (horizontal)
Ditto	Ditto				0.4%	3.4%	2.03 (vertical)

Table 1.5: Examples of the use of tuned dynamic dampers

Note: In the case of the Millenium footbridge, as well as ADAs, viscous dampers were also installed to dampen horizontal movement.

4. Footbridge dynamic analysis methodology

For the purposes of Owners, prime contractors and designers, this chapter puts forward a methodology and recommendations for taking into account the dynamic effects caused by pedestrian traffic on footbridges.
The recommendations are the result of the inventory described in Appendices 1 to 4, on the one hand, and of the work of the Sétra-AFGC group responsible for these guidelines, on the other. Thus, they are a summary of existing knowledge and present the choices made by the group.
The methodology proposed makes it possible to limit risks of resonance of the structure caused by pedestrian footsteps. Nevertheless, it must be remembered that, resonance apart, very light footbridges may undergo vibrational phenomena.
At source, when deciding on his approach, the Owner needs to define the class of the footbridge as a function of the level of traffic it will undergo and determine a comfort requirement level to fulfil.
Footbridge class conditions the need, or otherwise, to determine structure natural frequency. Where they are calculated, these natural frequencies lead to the selection of one or several dynamic load cases, as a function of the frequency value ranges. These load cases are defined to represent the various possible effects of pedestrian traffic. Treatment of the load cases provides the acceleration values undergone by the structure. The comfort level obtained can be qualified by the range comprising the values.
The methodology is summarised in the organisation chart below.

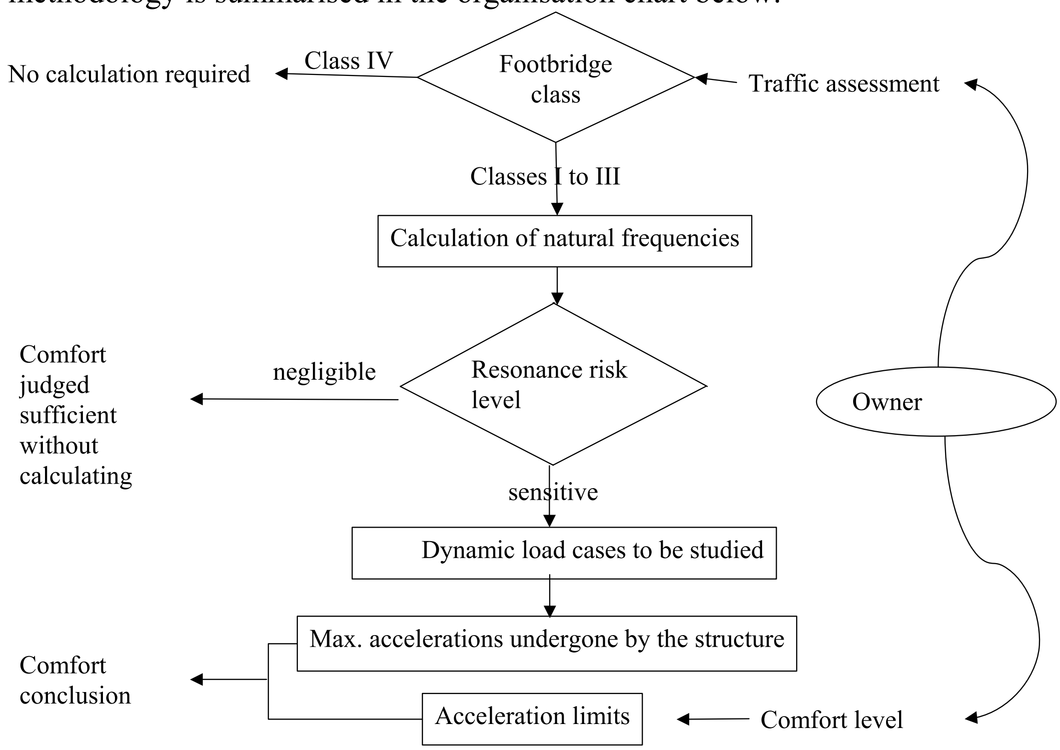


Figure 2.1: Methodology organisation chart

This chapter also includes the specific verifications to be carried out (SLS and ULS) to take into account the dynamic behaviour of footbridges under pedestrian loading (see 2.6). Obviously, the standard verifications (SLS and ULS) are to be carried out in compliance with the texts in force; they are not covered by these guidelines.

4.1 - Stage 1: determination of footbridge class

Footbridge class makes it possible to determine the level of traffic it can bear:
Class IV: seldom used footbridge, built to link sparsely populated areas or to ensure continuity of the pedestrian footpath in motorway or express lane areas.
Class III: footbridge for standard use, that may occasionally be crossed by large groups of people but that will never be loaded throughout its bearing area.
Class II: urban footbridge linking up populated areas, subjected to heavy traffic and that may occasionally be loaded throughout its bearing area.
Class I: urban footbridge linking up high pedestrian density areas (for instance, nearby presence of a rail or underground station) or that is frequently used by dense crowds (demonstrations, tourists, etc.), subjected to very heavy traffic.
It is for the Owner to determine the footbridge class as a function of the above information and taking into account the possible changes in traffic level over time.
His choice may also be influenced by other criteria that he decides to take into account. For instance, a higher class may be selected to increase the vibration prevention level, in view of high media expectations. On the other hand, a lower class may be accepted in order to limit construction costs or to ensure greater freedom of architectural design, bearing in mind that the risk related to selecting a lower class shall be limited to the possibility that, occasionally, when the structure is subjected to a load where traffic and intensity exceed current values, some people may feel uncomfortable.
Class IV footbridges are considered not to require any calculation to check dynamic behaviour. For very light footbridges, it seems advisable to select at least Class III to ensure a minimum amount of risk control. Indeed, a very light footbridge may present high accelerations without there necessarily being any resonance.

4.2 - Stage 2: Choice of comfort level by the Owner

4.2.1 - Definition of the comfort level

The Owner determines the comfort level to bestow on the footbridge.
Maximum comfort: Accelerations undergone by the structure are practically imperceptible to the users.
Average comfort: Accelerations undergone by the structure are merely perceptible to the users.
Minimum comfort: under loading configurations that seldom occur, accelerations undergone by the structure are perceived by the users, but do not become intolerable.
It should be noted that the above information cannot form absolute criteria: the concept of comfort is highly subjective and a particular acceleration level will be experienced differently, depending on the individual. Furthermore, these guidelines do not deal with comfort in premises either extensively or permanently occupied that some footbridges may undergo, over and above their pedestrian function.
Choice of comfort level is normally influenced by the population using the footbridge and by its level of importance. It is possible to be more demanding on behalf of particularly sensitive users (schoolchildren, elderly or disabled people), and more tolerant in case of short footbridges (short transit times).
In cases where the risk of resonance is considered negligible after calculating structure natural frequencies, comfort level is automatically considered sufficient.

4.2.2 - Acceleration ranges associated with comfort levels

The level of comfort achieved is assessed through reference to the acceleration undergone by the structure, determined through calculation, using different dynamic load cases. Thus, it is not directly a question of the acceleration perceived by the users of the structure.
Given the subjective nature of the comfort concept, it has been judged preferable to reason in terms of ranges rather than thresholds. Tables 2.1 and 2.2 define 4 value ranges, noted 1, 2, 3 and 4, for vertical and horizontal accelerations respectively. In ascending order, the first 3 correspond to the maximum, mean and minimum comfort levels described in the previous paragraph. The 4th range corresponds to uncomfortable acceleration levels that are not acceptable.

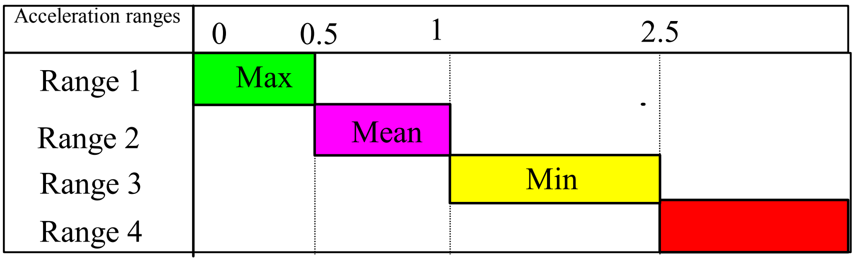


Table 2.1: Acceleration ranges (in m/s²) for vertical vibrations

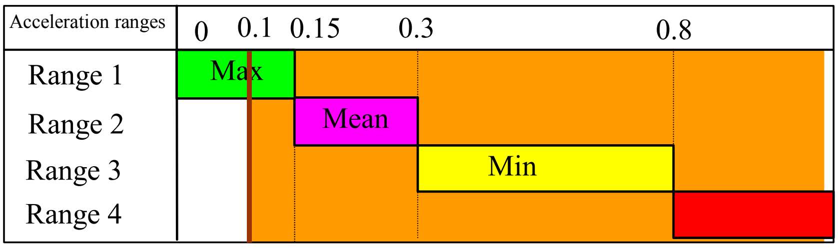


Table 2.2: Acceleration ranges (in m/s²) for horizontal vibrations

The acceleration is limited in any case to 0.10 m/s² to avoid "lock-in" effect

4.3 - Stage 3: Determination of frequencies and of the need to perform dynamic load case calculations or not

For Class I to III footbridges, it is necessary to determine the natural vibration frequency of the structure. These frequencies concern vibrations in each of 3 directions: vertical, transverse horizontal and longitudinal horizontal. They are determined for 2 mass assumptions: empty footbridge and footbridge loaded throughout its bearing area, to the tune of one 700 N pedestrian per square metre (70 kg/m^2).
The ranges in which these frequencies are situated make it possible to assess the risk of resonance entailed by pedestrian traffic and, as a function of this, the dynamic load cases to study in order to verify the comfort criteria.

4.3.1 - Frequency range classification

In both vertical and horizontal directions, there are four frequency ranges, corresponding to a decreasing risk of resonance:
Range 1: maximum risk of resonance.
Range 2: medium risk of resonance.
Range 3: low risk of resonance for standard loading situations.
Range 4: negligible risk of resonance.
Table 2.3 defines the frequency ranges for vertical vibrations and for longitudinal horizontal vibrations. Table 2.4 concerns transverse horizontal vibrations.

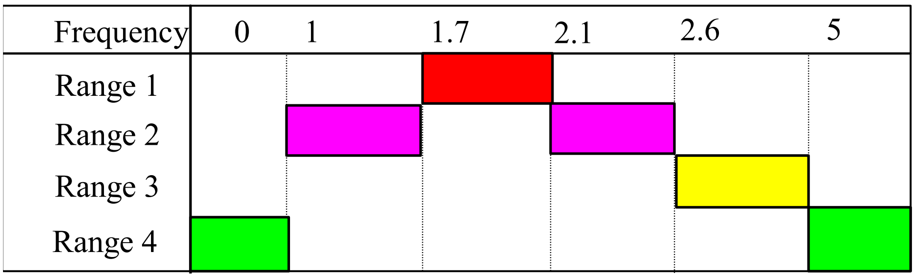


Table 2.3: Frequency ranges (Hz) of the vertical and longitudinal vibrations

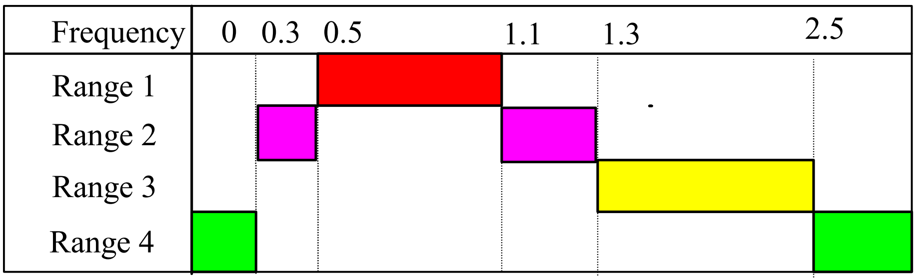


Table 2.4: Frequency ranges (Hz) of the transverse horizontal vibrations

4.3.2 - Definition of the required dynamic calculations

Depending on footbridge class and on the ranges within which its natural frequencies are situated, it is necessary to carry out dynamic structure calculations for all or part of a set of 3 load cases:
Case 1: sparse and dense crowd
Case 2: very dense crowd
Case 3: complement for an evenly distributed crowd (2nd harmonic effect)
Table 2.5 clearly defines the calculations to be performed in each case.

Traffic	Class	Load cases to select for acceleration checks – Natural frequency range 1	2	3
Sparse	III	Case1	Nil	Nil
Dense	II		Case 1	Case 3
Very dense	I	Case 2	Case 2	Case 3

Case No. 1: Sparse and dense crowd
Case No. 2: Very dense crowd
Case No. 3: Crowd complement (2nd harmonic)

Table 2.5: Verifications – load case under consideration:

4.4 - Stage 4 if necessary: calculation with dynamic load cases.

If the previous stage concludes that dynamic calculations are needed, these calculations shall enable:

- checking the comfort level criteria in paragraph II.2 required by the Owner, under working conditions, under the dynamic load cases as defined hereafter,
- traditional SLS and ULS type checks, including the dynamic load cases.

4.4.1 - Dynamic load cases

The load cases defined hereafter have been set out to represent, in a simplified and practicable way, the effects of fewer or more pedestrians on the footbridge. They have been constructed for each natural vibration mode, the frequency of which has been identified within a range of risk of resonance. Indications of the way these loads are to be taken into account and to be incorporated into structural calculation software, and the way the constructions are to be modelised are given in the next chapter.

Case 1: sparse and dense crowds

This case is only to be considered for category III (sparse crowd) and II (dense crowd) footbridges. The density d of the pedestrian crowd is to be considered according to the class of the footbridge:

Class	Density d of the crowd
III	0.5 pedestrians/m^2
II	0.8 pedestrians/m^2

This crowd is considered to be uniformly distributed over the total area of the footbridge S.
The number of pedestrians involved is therefore: $N = S \times d$.
The number of equivalent pedestrians, in other words the number of pedestrians who, being all at the same frequency and in phase, would produce the same effects as random pedestrians, in frequency and in phase is: $10{,}8 \times (\xi \times N)^{1/2}$ (see chapter 1).
The load that is to be taken into account is modified by a minus factor ψ which makes allowance for the fact that the risk of resonance in a footbridge becomes less likely the further

away from the range 1.7 Hz – 2.1 Hz for vertical accelerations, and 0.5 Hz – 1.1 Hz for horizontal accelerations. This factor falls to 0 when the footbridge frequency is less than 1 Hz for the vertical action and 0.3 Hz for the horizontal action. In the same way, beyond 2.6 Hz for the vertical action and 1.3 Hz for the horizontal action, the factor cancels itself out. In this case, however, the second harmonic of pedestrian walking must be examined.

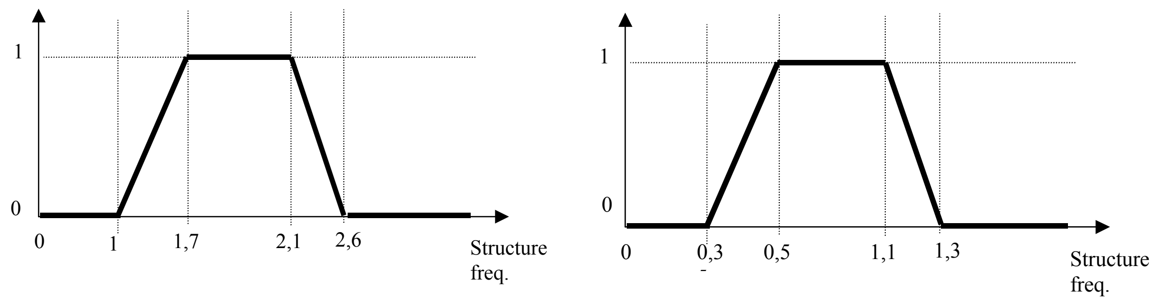


Figure 2.3 : Factor ψ in the case of walking, for vertical and longitudinal vibrations on the left, and for lateral vibrations on the right.

The table below summarises the load **per unit area** to be applied for each direction of vibration, for any random crowd, if one is interested in the vertical and longitudinal modes. ξ represents the critical damping ratio (no unit), and n the number of pedestrians on the footbridge (d x S).

Direction	Load per m²
Vertical (v)	$d \times (280N) \times \cos(2\pi f_v t) \times 10.8 \times (\xi / n)^{1/2} \times \psi$
Longitudinal (l)	$d \times (140N) \times \cos(2\pi f_l t) \times 10.8 \times (\xi / n)^{1/2} \times \psi$
Transversal (t)	$d \times (35N) \times \cos(2\pi f_v t) \times 10.8 \times (\xi / n)^{1/2} \times \psi$

The loads are to be applied to the whole of footbridge, and the sign of the vibration amplitude must, at any point, be selected to produce the maximum effect: the direction of application of the load must therefore be the same as the direction of the mode shape, and must be inverted each time the mode shape changes direction, when passing through a node for example (see chapter III for further details).

Comment 1: in order to obtain these values, the number of equivalent pedestrians is calculated using the formula $10.8 \times (\xi \times n)^{1/2}$, then divided by the loaded area S, which is replaced by n/d (reminder $n = S \times d$), which gives $d \times 10.8 \times (\xi / n)^{1/2}$, to be multiplied by the individual action of these equivalent pedestrians ($F_0 \cos(\omega t)$) and by the minus factor ψ.
Comment 2: It is very obvious that these load cases are not to be applied simultaneously. The vertical load case is applied for each vertical mode at risk, and the longitudinal load case for each longitudinal mode at risk, adjusting on each occasion the frequency of the load to the natural frequency concerned.
Comment 3: The load cases above do not show the static part of the action of pedestrians, G_0. This component has no influence on acceleration; however, it shall be borne in mind that the mass of each of the pedestrians must be incorporated within the mass of the footbridge.
Comment 4: These loads are to be applied until the maximum acceleration of the resonance is obtained. Remember that the number of equivalent pedestrians was constructed so as to compare real pedestrians with fewer fictitious pedestrians having perfect resonance. See chapter 3 for further details.

Case 2: very dense crowd

This load case is only to be taken into account for class I footbridges.
The pedestrian crowd density to be considered is set at 1 pedestrian per m^2. This crowd is considered to be uniformly distributed over an area *S* as previously defined.
It is considered that the pedestrians are all at the same frequency and have random phases. In this case, the number of pedestrians all in phase equivalent to the number of pedestrians in random phases (n) is 1.85 $\sqrt{n}$ (see chapter 1).
The second minus factor, ψ, because of the uncertainty of the coincidence between the frequency of stresses created by the crowd and the natural frequency of the construction, is defined by figure 2.3 according to the natural frequency of the mode under consideration, for vertical and longitudinal vibrations on the one hand, and transversal on the other.
The following table summarises the load to be applied per unit of area for each vibration direction. The same comments apply as those for the previous paragraph:

Direction	Load per m^2
Vertical (v)	$1.0 \times (280N) \times \cos(2\pi f_v t) \times 1.85\ (1/n)^{1/2} \times \psi$
Longitudinal (l)	$1.0 \times (140N) \times \cos(2\pi f_v t) \times 1.85\ (1/n)^{1/2} \times \psi$
Transversal (t)	$1.0 \times\ \ (35N) \times \cos(2\pi f_v t) \times 1.85\ (1/n)^{1/2} \times \psi$

Case 3: effect of the second harmonic of the crowd

This case is similar to cases 1 and 2, but considers the second harmonic of the stresses caused by pedestrians walking, located, on average, at double the frequency of the first harmonic. It is only to be taken into account for footbridges of categories I and II.
The density of the pedestrian crowd to be considered is 0.8 pedestrians per m^2 for category II, and 1.0 for category I.
This crowd is considered to be uniformly distributed. The individual force exerted by a pedestrian is reduced to 70N vertically, 7N transversally and 35N longitudinally.
For category II footbridges, allowance is made for the random character of the frequencies and of the pedestrian phases, as for load case No. 1.
For category I footbridges, allowance is made for the random character of the pedestrian phases only, as for load case No. 2.
The second minus factor, ψ, because of the uncertainty of the coincidence between the frequency of stresses created by the crowd and the natural frequency of the construction, is given by figure 2.4 according to the natural frequency of the mode under consideration, for vertical and longitudinal vibrations on the one hand, and transversal on the other.

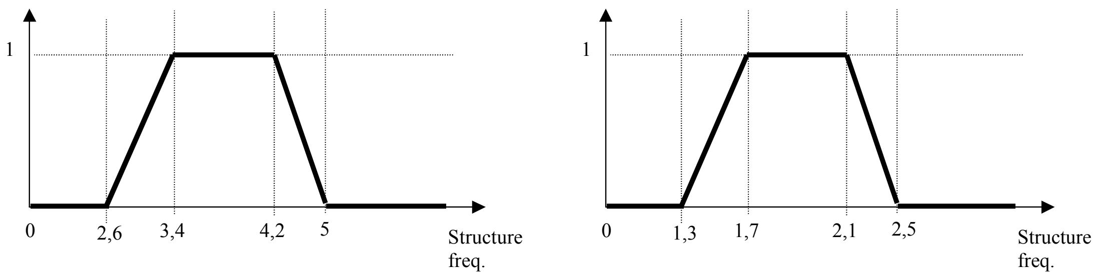


Figure 2.4: Factor ψ for the vertical vibrations on the left and the lateral vibrations on the right

4.4.2 - Damping of the construction

The dynamic calculations are made, taking into account the following structural damping:

Type	Critical damping ratio
Reinforced concrete	1.3%
Pre-stressed concrete	1%
Mixed	0.6%
Steel	0.4%
Timber	1%

Table 2.6: Critical damping ratio to be taken into account

In the case of different constructions combining several materials, the critical damping ratio to be taken into account may be taken as the average of the ratios of critical damping of the various materials weighted by their respective contribution in the overall rigidity in the mode under consideration:

$\xi_{\text{mode i}} = \dfrac{\sum_{material\ m} \xi_m k_{m,i}}{\sum_{material\ m} k_{m,i}}$ in which $k_{m,i}$ is the contribution of material m to the overall rigidity in mode i.

In practice, the determination of $k_{m,i}$ rigidity is difficult. For traditional footbridges of hardly varying section, the following formula approach can be used:

$\xi_{\text{mode}} = \dfrac{\sum_{material\ m} \xi_m EI_m}{\sum_{material\ m} EI_m}$ in which EI_m is the contribution of the material m to the overall rigidity EI of the section, in comparison to the mechanical centre of that section. (such that $\sum_{material\ m} EI_m = EI$)

4.5 - Stage 5: Modification of the project or of the footbridge

If the above calculations do not provide sufficient proof, the project is to be re-started if it concerns a new footbridge, or steps to be taken if it concerns an existing footbridge (installation or not of dampers). This paragraph gives recommendations for reducing the dynamic effects on footbridges. These recommendations are given in decreasing importance.

4.5.1 - Modification of the natural frequencies

The modification of the natural frequencies is the most sensible way of resolving vibration problems in a construction. However, in order to modify the natural frequencies of a construction significantly, it is very often necessary to carry out extensive structural modifications so as to increase the stiffness of the construction.

Most of the time, a way of increasing the natural frequencies is sought, so that the first mode, and thus all the following modes, are outside the range of risk. In certain cases, when the frequency of the first mode is low, but within the range of risk, and that of the second mode is sufficiently high, it may be advantageous to reduce the frequencies so as to bring the first mode below the range at risk, provided that the second mode remains above that range.

However, this is not very satisfactory. In addition, by reducing the rigidity of the construction, it becomes more flexible and static deflection is increased.
Of the ways of increasing the natural frequencies of the footbridge, the following can be quoted:

4.5.1.1 Vertical vibrations

Let us consider, for example, the case of the vertical vibrations of a deck formed from a steel box girder. If the depth of the box girder can be increased, its stiffness can thus be increased without increasing its mass. It is sufficient to retain the thicknesses of the top and bottom flanges and to reduce the thicknesses of the webs in proportion to the increase in their depth.
But, in a great number of cases, it is not possible to increase the depth of the box girder, or not by enough, for functional (height of passage under the footbridge) or architectural reasons. If the thicknesses of the flanges and of the webs of the box girder are increased, the inertia increases in proportion to the thickness, but the self-weight also increases, which reduces the overall effect. In this case there is no solution for increasing the natural frequencies, other than by modifying the static diagram of the construction: creating recesses in the piers, adding cable stays, etc.
In the case of a deck with a solid-core mixed steel-concrete beam, an increase in the thickness of the bottom steel flange is more effective in increasing the frequency of vibration. The self-weight does not increase as quickly, as it comprises a major part, due to the mass of the concrete slab, which does not increase.
In the case of a lattice deck, the inertia varies according to the square of the depth, whereas the section of the flanges (therefore the mass of the flanges) varies inversely to the depth. It is therefore advantageous to increase the depth in order to increase the frequency of vibration.
In the case of a concrete deck, an increase in the strength of the concrete enables its modulus, and thus the stiffness of the deck, to be increased, without increasing its mass, but in reduced proportions, as the modulus only increases according to the cube root of the compressive strength. Another traditional way of increasing stiffness without increasing the mass consists of replacing a rectangular section with an I-section. A deck formed by a box girder will thus have a higher frequency of vibration than a deck of the same thickness formed from rectangular beams.
Normal concrete can also be replaced by lightweight concrete in order to reduce the mass (with a slight reduction in stiffness) and thus increase the frequency of vibration.
In the case of a cable-stayed deck, an increase in the sections of the stays generally allows the stiffness to be increased without increasing the mass by much. This solution is effective, but it is not economical, as the quantity of stays has to be increased, with no offset. A fan arrangement of stays is stiffer than a harp arrangement. Taller pylons also lead to an increase in the stiffness without increasing the mass, and thus to an increase in the frequency of vibration.
In the case of a suspended deck, the frequency of vibration increases according to the square root of the tension of the cables divided by the linear density of the cables and of the deck. There is therefore no advantage in simply increasing the section of the cables. Their deflection must, in particular, be reduced.
The vertical stiffness of a deck can also be increased by making the balustrades participate in the stiffness.

4.5.1.2 Torsional vibrations

The torsional vibrations of the deck cause it to move vertically, away from the longitudinal axis of the construction. The values of the torsion vibration frequencies of the deck must also therefore be considered when the vertical vibrations felt by pedestrians are being studied. The

frequency of torsional vibration is proportional to the square root of the stiffness in torsion and inversely proportional to the square root of the polar density of the deck. There is therefore every advantage in designing a deck that is rigid in torsion.
There are several ways of increasing the frequency of the torsion vibrations of a deck. One of these, of course, is to increase the torsional inertia. A box girder deck thus has greater torsional inertia than a deck formed with lateral beams. The torsional inertia can be increased yet more by increasing the cross-sectional area of the box girder. The addition, to a deck formed from filler joists supported by lateral beams, of a bottom horizontal lattice wind-brace connecting the bottom flanges of the two beams, will also allow the torsional stiffness to be increased, but by a smaller amount than a box girder.
In the case of a cable-stayed bridge with lateral suspension, the deck of which is formed using two lateral beams, anchoring the cable stays into the axial plane of the construction (on an axial pylon or to the top of an inverted Y or inverted V pylon), and not into two independent lateral pylons, will allow the torsional frequency to be increased by a factor of close to 1.3 (Ref. [52]). This is what was done for the footbridge for the Palais de Justice in Lyon.

4.5.1.3 Horizontal vibrations

The frequency of horizontal vibration is proportional to the square root of the horizontal stiffness and inversely proportional to the square root of the mass of the deck.
One obvious way of increasing horizontal stiffness is to increase the width of the deck. But at a cost that is no less obvious.
At a given width, one way of increasing horizontal stiffness consists of providing resistant elements on the edges of the deck: for example, two S-section lateral beams can be used, rather than four S/2-section beams equally spaced under the deck.
In the case of cable-stayed or suspended footbridges that are very narrow in relation to their span, lateral cables can be used to stiffen the construction. This is the case for the suspended footbridge at Tours, over the Cher.

4.5.2 - Structural reduction of accelerations

If it is not possible to increase the frequencies sufficiently, or if the increase leads to a design that could make the project unviable, or if the bridge is an old one, to which substantial modification cannot be made, an attempt must be made to reduce accelerations (without directly affecting the damping). To do this, the mass of the construction can be increased by using a "heavy" deck (asphalt, concrete etc.). This has a direct effect on accelerations (it should be remembered that they are inversely proportional to the mass). In addition, if this deck is connected to the structure, the frequencies will not be reduced by too much, and the damping provided by the deck may make an appreciable contribution to the total damping.
Another way of reducing accelerations is to use materials that are naturally damping. However, it shall be realised that, for the damping of these materials to be mobilised, they must play a part in the overall rigidity. Increased damping may be obtained, for example, by the use of a lattice balustrade. In the case of "stress ribbon" footbridges, elastomer panels may be positioned between the pre-cast concrete slabs from which the deck is formed in order to increase the damping.

4.5.3 - Installation of dampers

As a last resort, if the previous solutions do not work, damping systems can be installed, which will most usually be tuned mass dampers (these are the easiest to install: to work properly, viscous dampers often require the construction of complex devices to recreate major differential movement). A tuned mass damper consists of a mass connected to the construction using a spring, with a damper positioned in parallel. This device allows the vibrations in a construction to be reduced by a large amount in a given vibration mode, under the action of a periodic excitation of a frequency close to the natural frequency of this vibration mode of the construction (see Annexe 3 §3.3).

This shall only be considered as a last resort, as, despite the apparently attractive character of these solutions (substantial increase in damping at low cost), there are disadvantages. If tuned mass dampers are used, which is the most typical case:

- as many dampers are needed as there are frequencies of risk. For complex footbridges, which have many modes (bending, torsion, vertical, transversal, longitudinal modes, etc.) of risk, it may be very onerous to implement;
- the damper must be set (within about 2-3%) at a frequency of the construction that changes over time (deferred phenomena) or according to the number of pedestrians (modification of the mass). The reduction in effectiveness is appreciable;
- the addition of a damper degenerates, and thus doubles, the natural frequency under consideration: this complicates the overall dynamic behaviour, and also the measurement of the natural frequencies;
- even though manufacturers claim that dampers have a very long life-span, they do need a minimum level of routine maintenance: Owners must be made aware of this;
- because of the added weight (approximately 3 to 5% of the modal mass of the mode under consideration), this solution will only work on an existing footbridge if it has sufficient spare design capacity. On a proposed footbridge, the designer may need to resize the construction;
- preferably, 3% of guaranteed damping will be achieved: on very lightweight constructions (for which the ratio of the exciting force divided by the mass is high), it may not be sufficient.

4.6 - Structural checks under dynamic loads

4.6.1 - SLS type checks specific to the dynamic behaviour

In addition to the usual checks to be made on service limit states, as defined by the relevant regulations, specific service combinations for the dynamic loads of pedestrians should be built up, by combining:

- the effects (stresses, displacements, etc.) resulting from one of the dynamic load cases 1 to 4, by applying the same methodology as for the determination of comfort. Thus, when accelerations have to be calculated, the corresponding forces must also be calculated to check whether they remain permissible. If the methodology does not require the calculation of accelerations, this check need not be made;
- the effects of static loading associated with the dynamic case under consideration, corresponding on the one hand to the permanent loads and on the other hand to the weight of the pedestrians set out identically, using a load of 700 N per pedestrian.

It is important to note that static and dynamic vibratory shapes are combined, and, therefore, that precautions must be taken when using calculation software.

4.6.2 - ULS type checks specific to the dynamic behaviour

As these consist of checking the strength of the construction, using levels of stresses that are assumed to be close to the elastic limit of the materials (otherwise, obviously, there would be no strength problem), the dynamic calculations must be made using the structural damping given in table 2.7 below:

Types of construction	Damping
welded steel	2%
bolted steel	4%
reinforced concrete	5%
pre-stressed concrete	2%

Table 2.7

In the same way as for service limit states, the ultimate limit states set out here should be carried out in addition to the traditional ULSs imposed by the relevant regulations. An accidental type combination should be formed in order to simulate a case of vandalism or an exceptionally large public demonstration on the footbridge. The check is to be made as soon as a comfort check for a natural frequency within a range at risk has been made. The load case to be considered is similar to load case No. 1 previously defined (see § 2.3.2), whatever the category of the footbridge, with the following modifications:

- the density of the crowd is taken as 1 pedestrian per m²;
- apart from the permanent loads on the footbridge, allowance must be made for the static load applied by the crowd, which is taken as 700 N/m²;
- the individual effects of pedestrians are combined directly, with no minus factor ($n_{equivalent} = n$ and $\psi = 1$).

This load case is extremely pessimistic, as it assumes the perfect coordination of a whole crowd of pedestrians. However, it could occur under very exceptional conditions (rhythmic demonstrations, races, processions, etc.) or under lock-in situations.

Experience has shown that there may be a problem of strength in particular cases only, as it must not be forgotten that such a load case represents an accidental ULS, for which the permanent loads can be weighted differently from the fundamental ULS, or can be more tolerant on the limits of the materials.

Whatever, there may, under certain particular circumstances, be a lack of strength if the construction is optimised and everything precisely dimensioned. If such is the case, and if the consequences on sizing are major (which should not be the case, due to the small difference between sizing with and without dynamic loads), there may be an advantage in looking to which acceleration the maximum dynamic force thus calculated corresponds. If this acceleration is too great, it could be given a reasonable value less than the acceleration due to gravity (between 0.5 g and 1.0 g), beyond which it can be considered that walking becomes impossible.

In the transverse direction, the strength of the construction is not very likely to be a problem, as:

- The lateral action of a pedestrian is much weaker than the vertical action (35 N instead of 280 N);
- The consequences on sizing may be negligible because of the sizing of the footbridge for wind, assuming that extreme wind situations and a large crowd on the footbridge are not combined;

- The transverse acceleration beyond which a pedestrian can no longer keep his balance is much smaller than the vertical acceleration. It can, for example, be limited to a value of between 0.1 g and 0.3 g.

It is therefore fairly unlikely that an exceptional pedestrian crowd could cause a problem for the sizing of a pedestrian footbridge, or of its strength, but a check should first be made.

Finally, it should be noted that, in these checks, the individual safety of the pedestrians in the event of accelerations that are too great because of these exceptional public demonstrations has not been covered. That is why this guide recommends Owners not to permit the holding of exceptional public demonstrations on pedestrian footbridges, but to take them into account in the sizing of the footbridge, as it may be difficult to prevent them from taking place throughout the lifespan of the footbridge.

5. Chapter 3: Practical design methods

5.1 - Practical design methods

For the dynamic design of continuous constructions and systems, two major methods can be used: direct integration design and modal design.

5.1.1 - Direct integration

This method consists of integrating the equations of the dynamics directly, with an imposed load. It is used very little in practice for vibrations of footbridges, as the phenomena affecting them are resonance phenomena, which means that, to predict them, the natural frequencies of the constructions must be known. It is used more in earthquake-resistant design, in which the excitation is known (imposed earthquake accelerogram, for example)
The direct integration method, more expensive in analysis time than modal calculations, may, however, turn out to be necessary in one of the following cases:

- when the modal superposition method cannot be used with a reduced number of modes;
- when the damping is not proportional or when it is concentrated (viscous dampers for example);
- when the problem is not linear.

The two latter points may be encountered in certain footbridge vibration problems (use of finite elements, dampers, construction with non-linear behaviour). There do exist, however, methods that are close to modal methods, and it is often advantageous, even so, to determine "apparent frequencies" of normal vibrations in order to be able to set the dynamic loads producing the maximum effect.

5.1.2 - Modal calculation

The modal method is always carried out in two stages, namely the determination of the natural vibration modes initially, and the calculation of the actual response (if necessary) on the basis formed by these natural vibration modes.
The power of this method is that the natural vibration modes are the natural vibration modes of the construction. They therefore represent the most likely vibration modes of the construction.
The second advantage is that they form a base, and that the actual solution is therefore a linear combination of these modes. The number of degrees of freedom is thus considerably reduced and non-linked equations are obtained.
Finally, the third advantage is that, when a construction is excited at one of its natural frequencies, and only in this case, a resonance phenomenon is produced. One of the modes then has a much larger response than the others. A problem with several degrees of freedom is thus restricted to a problem with one degree of freedom, thus easy to resolve. There is, therefore, then only one unknown in the problem, the amplitude of that resonance. A knowledge of these resonance frequencies is crucial for the analysis of the dynamic problem.

5.2 - Dynamic calculation applied to footbridges

5.2.1 - Calculation of natural frequencies and natural vibration modes

In practice, the calculation of the natural vibration modes is made by using software for complex constructions (which is achieved, in practice, as soon as there are several beams, etc.). However, a certain number of situations may be determined analytically. It is also often possible to have orders of magnitude of natural frequencies, even in the case of complex constructions.

5.2.1.1 Simply-supported beam

For a simply-supported beam of constant characteristics, an analytical calculation is possible (see table 3.1).

Mode	Natural pulsation	Natural frequency	Mode shape
Simple bending to n sags	$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho\ S}}$	$f_n = \frac{n^2\pi}{2L^2}\sqrt{\frac{EI}{\rho\ S}}$	$v_n(x) = \sin\left(\frac{n\pi x}{L}\right)$
Tension-compression to n sags	$\omega_n = \frac{n\pi}{L}\sqrt{\frac{E\,S}{\rho\ S}}$	$f_n = \frac{n}{2L}\sqrt{\frac{E\,S}{\rho\ S}}$	$u_n(x) = \sin\left(\frac{n\pi x}{L}\right)$
Torsion to n sags	$\omega_n = \frac{n\pi}{L}\sqrt{\frac{GJ}{\rho\ I_r}}$	$f_n = \frac{n}{2L}\sqrt{\frac{GJ}{\rho\ I_r}}$	$\theta_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

Table 3.1: analytical modes for a simply-supported beam

ρS is the linear density of the construction (including the permanent loads and the varying loads), ρI_r is the moment of inertia of the construction, *ES* the tension-compression rigidity, *EI* the bending rigidity, *GJ* the torsional rigidity.

In practice, for footbridges that are not very wide (low width in comparison with span) and rigid in torsion (closed profiles), the torsion modes are at high frequencies, as are the tension-compression modes. In this case, only an analysis of the bending modes is pertinent. If the sections are flexible in torsion (case of open profiles), the torsion modes must be taken into account. If the sections are wide (slabs or double beams, for example), allowance must also be made for the differential bending or transverse bending modes. (See figure 3.1). In such a case, "plate and shell" type modelisations may turn out to be necessary.

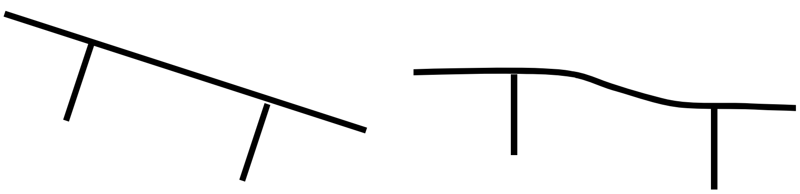
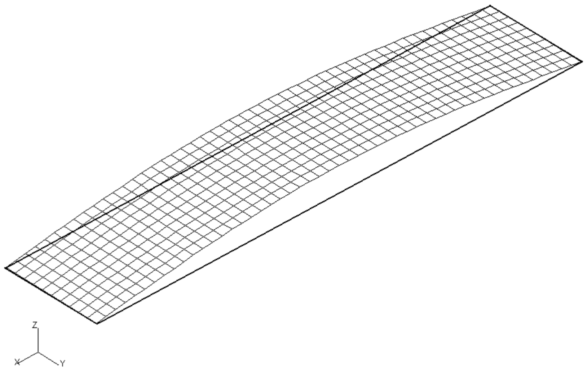
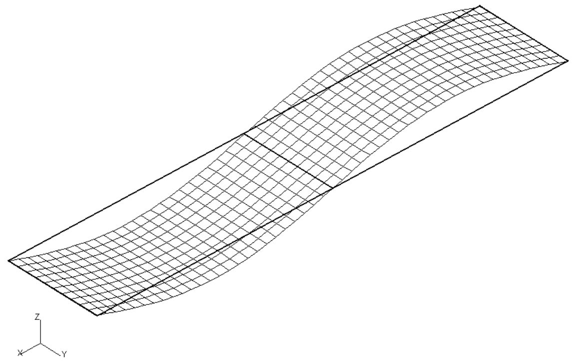


Figure 3.1: Torsion mode (on the left) and differential bending mode (on the right) to be taken into account when the profiles are open (on the left), or when the sections are wide (on the right), or even in both cases.

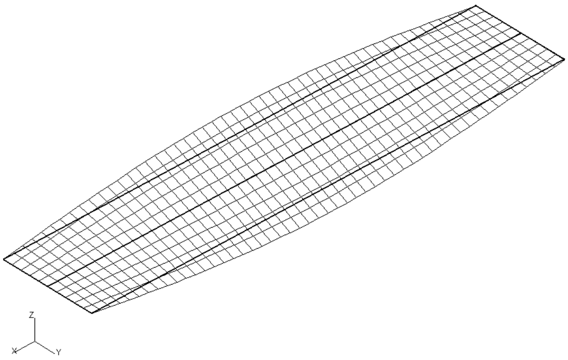
The following figures 3.2 illustrate the bending and torsion modes encountered on a footbridge with a width one-fifth of the span.



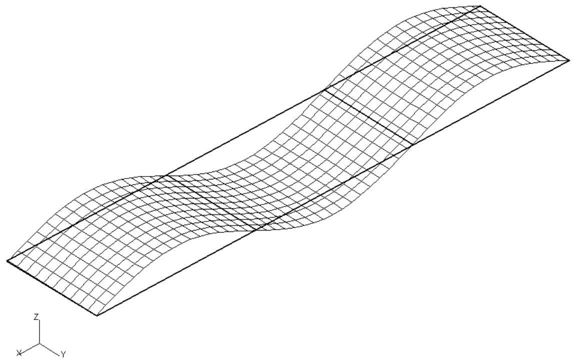
Mode 1: Bending 1 sag



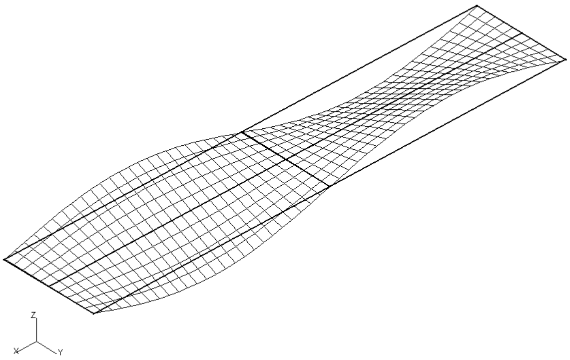
Mode 2: Bending 2 sags



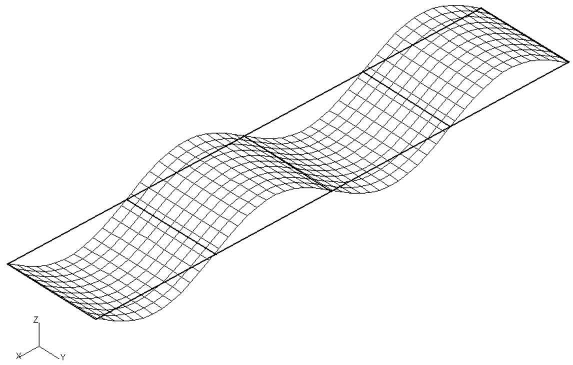
Mode 3: Torsion 1 sag



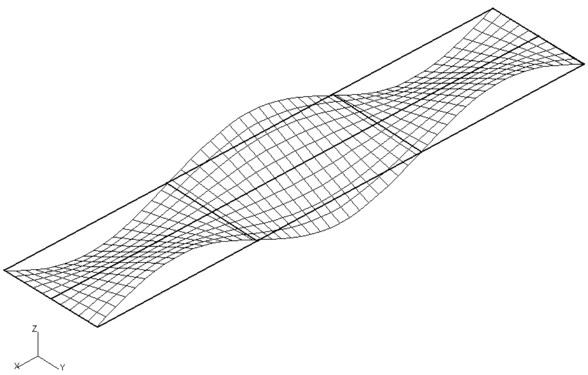
Mode 4: Bending 3 sags



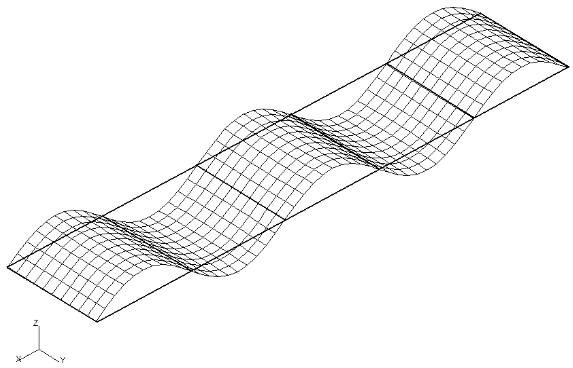
Mode 5: Torsion 2 sags



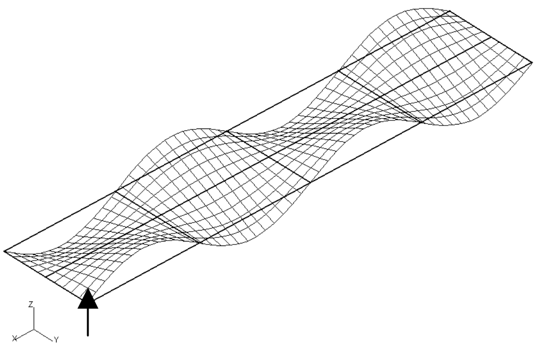
Mode 6: Bending 4 sags



Mode 7: Torsion 3 sags



Mode 8: Bending 5 sags



Mode 9: Torsion 4 sags

Figure 3.2 Mode shape of a wide simply-supported footbridge

5.2.1.2 Simple beam with various boundary conditions

For any beam, continuous over supports, with different boundary conditions, the natural pulsations are always in the form $\omega_n = \frac{\lambda_n}{L^2}\sqrt{\frac{EI}{\rho S}}$ which can be broken down into one factor depending on the shape of the beam λ_n, one factor depending on the material $\sqrt{\frac{E}{\rho}}$ and one factor depending on the section $\sqrt{\frac{I}{S}}$, also known as turning radius.

The following table, taken from the CECM 89 rules (Ref. [8]), gives the values of the factor λ_n depending on the support conditions:

r = 1	r = 2	r = 3	r = 4	r = 5	r > 5
λ = 3,52	0,774 λ = 22,4	0,500 0,868 λ = 61,7	0,644 0,356 0,906 λ = 121,0	0,500 0,928 0,279 0,723 λ = 200,0	$[(2r-1)\frac{\pi}{2}]^2$
λ = 9,87	0.500 λ = 39.5	0.333 0,667 λ = 88,9	0,50 0.25 0.75 λ = 158	0.20 0.40 0.60 0.80 λ = 247	$[r\ \pi]^2$
λ = 22,4	0.500 λ = 61,7	0.339 0.641 λ = 121	0.500 0.259 0.772 λ = 200	0.409 0.773 0.227 0.591 λ = 298	$[(2r+1)\frac{\pi}{2}]^2$
0.224 0.776 λ = 22,4	0.132 0.500 0.860 λ = 61,7	0.094 0.644 0.356 0.906 λ = 121	0.277 0.723 0.073 0.500 0.927 λ = 200	0.060 0.409 0.773 0.221 0.591 0.940 λ = 298	$[(2r+1)\frac{\pi}{2}]^2$
λ = 15,4	0.560 λ = 50,0	0.384 0.692 λ = 104	0.529 0.294 0.765 λ = 178	0.429 0.910 0.230 0.519 λ = 272	$[(4r+1)\frac{\pi}{4}]^2$
0.736 λ = 15,4	0.440 0.953 λ = 50,0	0.616 0.309 0.893 λ = 104	0.471 0.922 0.235 0.707 λ = 178	0.381 0.763 0.130 0.591 0.937 λ = 272	$[(4r+1)\frac{\pi}{4}]^2$

Table 3.2: Influence of the boundary conditions on the natural vibration modes

5.2.1.3 Commonplace constructions

Before carrying out any digital calculation of the dynamics of a footbridge, whether for simple cases using frequency and acceleration calculation formulae, or for the modelling of more complex constructions on particular calculation programs, a certain number of questions should first be asked on the calculation assumptions and the model being created.
The following list is intended to set out the main points that have to be watched carefully to do the modelling successfully. These points cover the dynamic calculation itself. Obviously, all the usual static structure modelling assumptions and techniques should be carried out. This concerns, for example, the relevance of the finite element grid, and thinking about the modelling of links in steel constructions, etc.
The modeliser must take particular care with the following aspects and questions before starting the digital modelling of the construction:

5.2.1.3.1 Structure of the deck

Is the deck topping participating or not? If not, only its mass is relevant, if yes, it will, in addition, provide rigidity which will have a direct effect on frequencies, and damping, which will have a direct effect on the overall structural damping.

The links between the various elements must be described and taken into account: swivel joints, recesses, intermediate connections will have a direct effect on frequencies.
Welds or bolted connections lead to different damping.

5.2.1.3.2 Supports

Particular attention shall be paid to supports. The dynamic behaviour of the structure of a footbridge deck is often taken into account, but not that of its supports. In most typical cases, this simplification is permissible, as the supports are often more rigid than the footbridge deck and provide more damping. But this is not always the case.
At the base of the support, there is the foundation, on piles or a strip footing. This foundation has certain flexibility linked to that of the ground. In general, it also provides a high level of damping, due to the friction between the ground and the construction. In certain cases, either its horizontal and/or its vertical components may be taken into account.

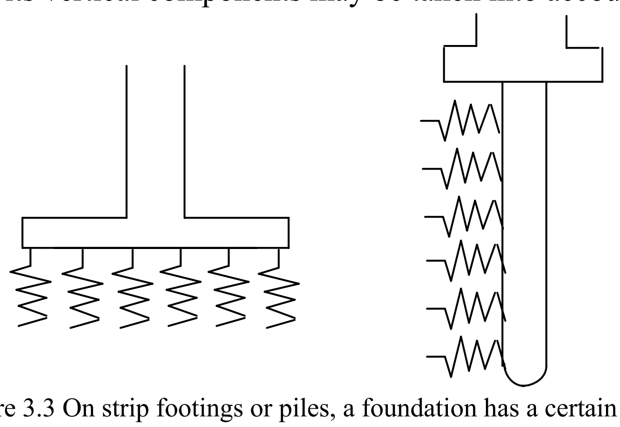


Figure 3.3 On strip footings or piles, a foundation has a certain rigidity k_{sol}

Then there is the pier. If this is very high, its flexibility becomes important and it may affect the natural frequencies of the overall construction.

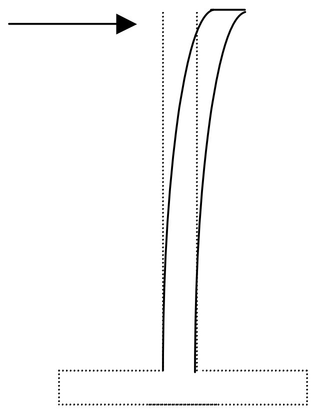


Figure 3.4: Very high pier: The rigidity of this type of support, held at its base and free at its top may be evaluated as $k_{pile} = 3EI/h^3$ in which h is the height, I the inertia and E the elastic modulus.

Finally, the support fittings must not be forgotten, nor their effect on the overall rigidity of the construction and on overall damping.

For a support fitting in reinforced elastomer, size $a \times a$, the thickness of one layer of which is e, with a shear modulus G and with n layers, the horizontal stiffness is:
$k_{support\ fitting} = G\, a^2 / n\, e$
All these intermediate elements may modify the natural frequencies of the footbridge and their associated damping. The theoretical model and simplified diagram may become as follows:

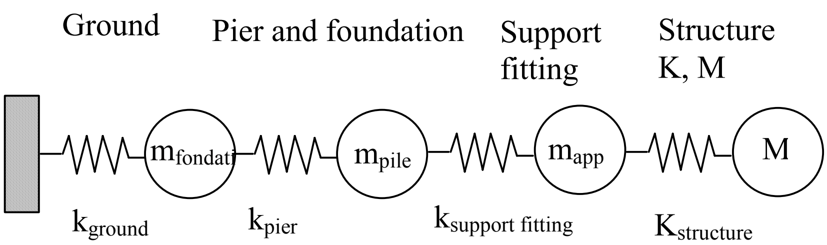


Figure 3.5: "Simplified" diagram of the construction down to the foundations

In the case of continuous systems, there is not necessarily a single stiffness for each of the above elements. A complete model, often digital, will then be necessary.

5.2.1.3.3 Making allowance for the mass

Certain dynamic calculation software remains confused as to making allowance for mass. It is an important factor that must be properly dealt with by the software by distinguishing it from the static effects of self-weight. Care must be taken for example, not to confuse rotation inertia, useful for the calculation of the moment of inertia, and torsion inertia, useful for the calculation of rigidity in torsion.
In addition, as the mass has a direct influence on the frequencies of the modes, the mass of the pedestrians, either standing still or walking on the footbridge, shall be taken into account.
Thus, for each mode, the frequency of the mode in question is located within a varying range between two extreme frequencies f_1 and f_2, one calculated with the mass of the construction and of the pedestrians (f_1) and the other with the mass of the construction only (f_2). For very lightweight footbridges, these two frequencies may be some distance apart.

They should, therefore, both be calculated. In the same way, when calculating accelerations, there must be coherence between the load case carried out and the mass taken into account (and several calculations made if necessary).

5.2.1.3.4 Making allowance for dampers

The dynamic calculation may be carried out with an allowance for tuned mass dampers (in the case of a recalculation of an existing footbridge that has to be upgraded by the addition of dampers, or in the case of a footbridge of a design that would not allow the saving by using dampers to be made).
These dampers may be incorporated directly into the digital model if the model allows them to be modelised. As, in most cases, it is a question of adding a mass, spring, damper system, design software generally allows this to be done easily.
The analytical calculation may be made mode by mode if the frequencies of the footbridge are known, and also its modal masses and modal stiffness. In this case, it is possible to calculate the natural frequencies of the following system for each of the modes:

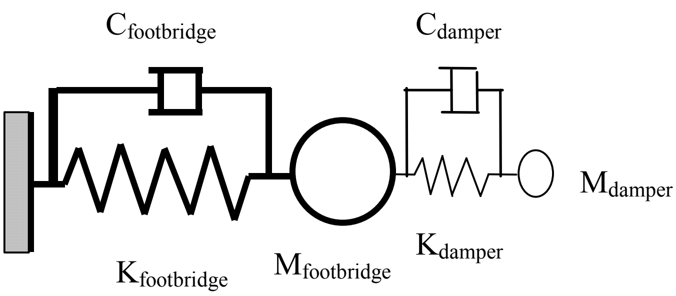


Figure 3.6: Schematic representation of a Construction – Damper system

$M_{footbridge}$ is the generalised mass of the mode of the footbridge under consideration, $K_{footbridge}$ the generalised stiffness of the footbridge mode under consideration and $C_{footbridge}$ its generalised damping. Annexe 3 gives the analytical solution of the dynamic amplification in accordance with the relationship between the exciting frequency and the initial frequency of the footbridge. This graph reveals the two natural vibration modes thus degenerated and their associated amplifications (thus the damping).

5.2.1.3.5 Particular features

The following points, likely to modify the dynamic model of the footbridge, shall, finally, be taken into account:

- existence of an initial constraint that influences the natural frequencies (pre-stressing in pre-stressed concrete, pre-tension in cable stays or ties, thrust from arches, etc.);
- non-linearity due to the cables or ties, materials, large displacements, pendulum effects, etc.;
- distinction between local modes (for example, vibration of a slab unit on a footbridge, of no danger to a crowd of pedestrians) and global modes;
- ***dynamic*** characteristics of the materials to be taken into account (concrete, stiffness of the ground, etc.).

5.2.2 - Practical calculation of the loading response

5.2.2.1 Principle

Remember the equation governing the behaviour of the mode *i* with modal damping (see annexe 1):

$$\ddot{q}_i(t) + 2\xi_i \omega_i \dot{q}_i(t) + \omega_i^2 q_i(t) = p_i(t) = \frac{{}^t[\phi_i][F(t)]}{m_i}$$

Each of these responses q_i is calculated separately from the others, then the global response is deducted by recomposing the modes. If the function $[F(t)]$ is harmonic ($[F(t)] = [F_0]\sin(\omega t)$), at the frequency of one of the modes (mode j for example), there is then resonance of that mode. The response q_j of mode j is much greater than the others and the global response, after a transitory period, is close to:

$$[X(t)] = \sum_{i=1}^{n} [\phi_i] q_i(t) \approx [\phi_j] q_j(t)$$

The amplitude of the dynamic response $q_j(t)$, ***after the transitory period,*** is obtained by multiplying the static response (that obtained as if the load was constant and equal to $[F_0]$, in other words that: $\omega_j^2 q_{j\,static} = \frac{{}^t[\phi_j][F_0]}{m_j}$) by the dynamic amplification factor $\frac{1}{2\xi_j}$.

The response obtained is therefore:

Displacement: $q_j(t) = -\frac{1}{2\xi_j}\frac{1}{\omega_j^2}\frac{{}^t[\phi_j][F_0]}{m_j}\cos(\omega_j t) +$ transitory response

Acceleration: $\ddot{q}_j(t) = \frac{1}{2\xi_j}\frac{{}^t[\phi_j][F_0]}{m_j}\cos(\omega_j t) +$ transitory response

It can be seen that the displacement and the acceleration are out of phase by a quarter of the natural period in comparison with the excitation. The speed is in phase with the excitation.
The global displacement response can be written:

$$[X(t)] = [X]_{static} + q_j(t)[\phi_j]$$

and in acceleration:

$$[\ddot{X}(t)] = \ddot{q}_j(t)[\phi_j]$$

Remember that the various calculations set out in this guide are based on the notion of number of equivalent pedestrians, which is the number of fictitious pedestrians that are all in phase, at the natural frequency of the footbridge, regularly spaced, whose action is in the same direction as the mode shape at each point, and, especially, at the maximum resonance, in other words in a steady state. As a result, the transitory response is not interesting and only the amplitude of the permanent response $\frac{1}{2\xi_j}\frac{{}^t[\phi_j][F_0]}{m_j}[\phi_j]$ is interesting, with $m_j = {}^t[\phi_j][M][\phi_j]$

5.2.2.2 Positioning of the load

As specified in the previous chapter, the load must be such that the amplitude of the force is of the same sign as the mode shape.

In the case of modes with several sags in the longitudinal or transverse direction, this means that the amplitude of the force must have the shape shown in figure 3.7:

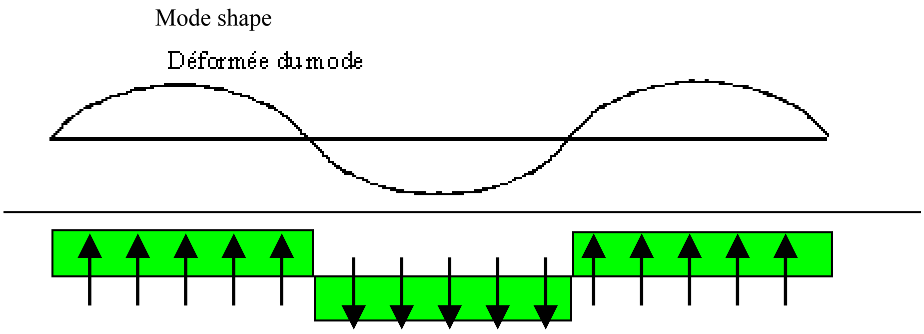


Figure 3.7: Sign of the amplitude of the load in the case of a mode with several sags

In the case of torsion modes with several sags, the amplitude of the force must be of the shape shown in figure 3.8

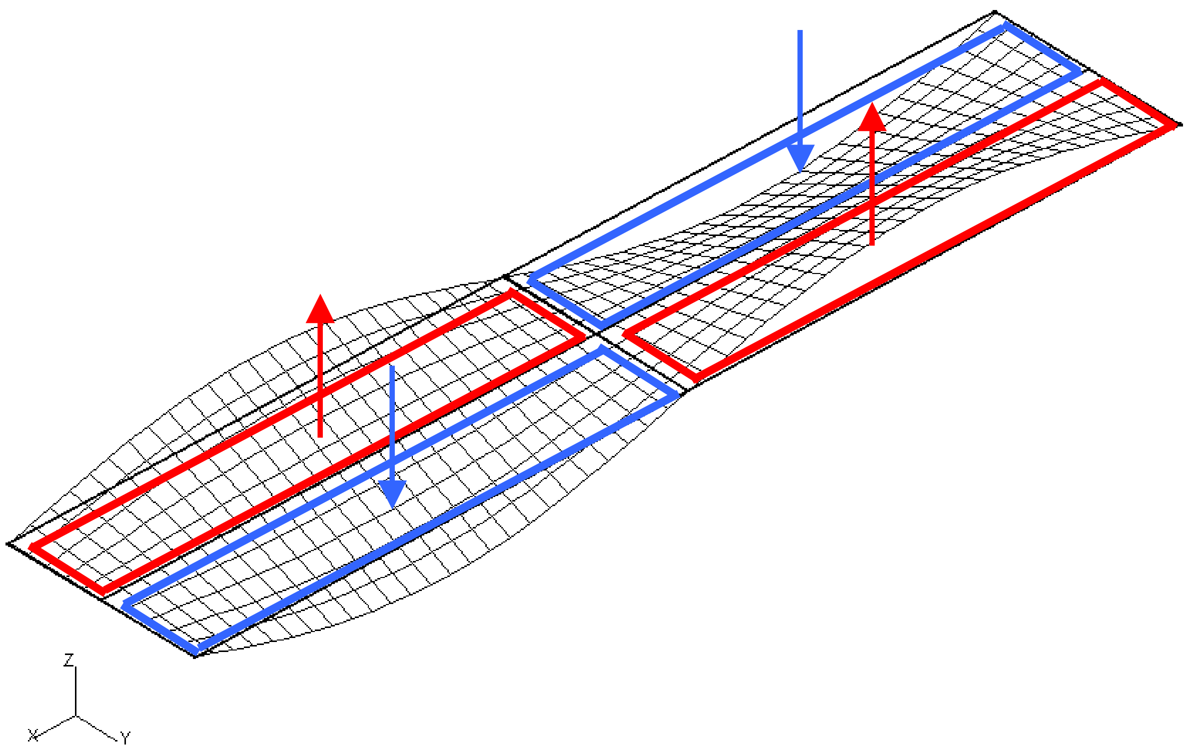


Figure 3.8: Sign of the amplitude of the load in the case of a torsion mode with several sags. Noted in red are the zones in which the amplitude of the load is positive, and in blue the zones in which the amplitude of the load is negative.

5.2.2.3 Practical calculation in the general case

In the general case, dynamic calculation software must be used to evaluate the natural frequencies, and also the accelerations obtained with the various loads. Several programs are available that take the dynamic calculation into account. Although most of them allow a calculation of the natural vibration modes and natural frequencies, there are fewer that allow the temporal calculation of the accelerations under varying loads. This is because these programs, that make calculations dynamically, are, in practice, intended especially for seismic calculations, and because, in such calculations, it is, in general, enough to calculate the natural vibration modes, and the application of standardised response spectra s that avoids the need for any real dynamic calculation.

In the case of pedestrian footbridges, these programs aimed at seismic design are not necessarily relevant. However, the results that they provide can be used to calculate the maximum accelerations, in the way explained below:

5.2.2.3.1 Programs calculating only the natural vibration modes and modal characteristics.

With this type of program, the modal information should be recovered for each natural vibration mode within the range at risk: vibratory shape at any point of the mode, natural frequency, modal damping (if possible), generalised mass. Care must be taken with the notion of generalised mass, which is different from modal mass, often given by the programs. The generalised mass is the value $m_j = {}^t[\phi_j][M][\phi_j]$ which is used in the modal equation of the dynamics and depends on the standardisation that has been selected (its final result is, however, independent), while the modal mass, the expression of which is $\frac{\left({}^t[\phi_j][M][\Delta]\right)^2}{{}^t[\phi_j][M][\phi_j]}$, is a value used typically in seismic calculations to evaluate the number of modes to be taken into account to represent properly the response to a seismic stress. In practice, it enables the natural vibration modes of low modal mass in relation to the mass of the construction to be eliminated. But the modal mass has no physical reality in relation to the generalised mass of the mode that is a proper representation of the vibrating mass in the mode under consideration, in the light of the selected standard.

In practice, for the mode j under consideration, the program must be asked to provide, for each grid i of the model, the values of the displacements of the mode shape V_{ij}, together with the value of the natural frequency. The generalised mass can then be formed by $m_j = \sum_i M_i V_{ij}^2$ in which M_i is the mass of the grid i.

The modal load is then formed, the projection of the load onto the mode, by writing $f_j = \sum_i F_i V_{ij}$ in which F_i represents the value of the force on the grid i. It is written $F_i = \overline{F}_i \cos(\omega t)$. Because of the loading mode of the beams, $\overline{F}_i$ has the same sign as V_{ij}. Its value is given in chapter 2. The second member of the modal equation is then written $\frac{f_j}{m_j}$.

The amplitude of the acceleration response, at the resonance, is then written simply, at the position of grid k: $\frac{1}{2\xi_j}\frac{f_j}{m_j}V_{kj} = \frac{1}{2\xi_j}\frac{\sum_i \overline{F}_i V_{ij}}{\sum_i M_i V_{ij}^2}V_{kj}$ (value independent of the standard selected for the modes. Indeed, if every V_{ij} is multiplied by the same constant, the result does not change)

5.2.2.3.2 Programs enabling temporal dynamic calculations.

With programs enabling temporal dynamic calculations, it is not necessary to use the tricks set out above.

One can either define a load that varies over time in the form $q\cos(\omega t)$, having previously determined exactly the natural frequencies, band looking at the dynamic response at the end

of a sufficiently long length of time (in practice, the time at the end of which the amplitude of the accelerations becomes just about constant). On the other hand, one must ensure that the frequency of the load is exactly equal to the natural frequency of the construction, which is a risk for the calculation.
The second method, easier to implement and carrying less risk, consists of selecting for the dynamic calculation only the mode under consideration (it must be possible to deactivate the other modes), and then apply to it the same load as previously, but assumed to be constant (therefore q alone, without the temporal section) which is easier to define. Then the displacement obtained at the end of a sufficiently long length of time is determined (in other words when the vibrations have smoothed out) and multiplied by $\omega_j^2 \frac{1}{2\xi_j}$ in order to give the acceleration at the resonance of the point under consideration.

5.2.2.4 Theoretical calculation for a simply supported beam

For a footbridge with constant inertia on two supports, the characteristic values of the footbridge can be easily calculated, as shown in table 3.3.

Type of value	Literal expression
Natural pulsations	$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho S}}$
Natural frequencies	$f_n = \frac{n^2\pi}{2L^2}\sqrt{\frac{EI}{\rho S}}$
Maximum deflection	$v_{\max} = \frac{1}{2\xi_n}\frac{1}{n^4}\frac{4FL^4}{EI\pi^5}$
Maximum moment	$M_{\max} = \frac{1}{2\xi_n}\frac{1}{n^2}\frac{4FL^2}{\pi^3}$
Maximum shear force	$V_{\max} = \frac{1}{2\xi_n}\frac{1}{n}\frac{4FL}{\pi^2}$
Maximum acceleration	$\text{Acceleration}_{\max} = \frac{1}{2\xi_n}\frac{4F}{\pi\rho S}$

Table 3.3: modal values for a simply-supported beam subjected to a load on a footbridge depending on the mode with *n* sags. *F* represents the amplitude of the force per unit length.

It can be noted that, if the exciting frequency ω is not exactly equal to the natural frequency ω_n, the term $\frac{1}{2\xi_n}$ is replaced by the dynamic amplification $A(\Omega) = \frac{1}{\sqrt{(1-\Omega^2)^2 + 4\xi^2\Omega^2}}$ (with $\Omega=\omega/\omega_n$).

6. Chapter 4: Design and works specification, testing

6.1 - Examples of items for a footbridge dynamic design specification

The specification shall state that the structural design of the footbridge shall be carried out in accordance with the recommendations in the SETRA / AFGC guide on the dynamic behaviour of pedestrian footbridges.
The Owner shall, however, specify:

- The category of the footbridge (I to IV)
- The acceptable comfort level (maximum, medium, minimum)

6.2 - Examples of items to be inserted in the particular works technical specification for a footbridge

It is recommended that this guide be annexed to the technical "works" specification of a footbridge and to state that the working design must follow the recommendations in this guide. The following shall, therefore, be stated once again:

- The category of the footbridge (I to IV)
- The acceptable comfort level (maximum, medium, minimum)

In addition, if studies have shown that the dynamic behaviour can only be ensured by the use of dampers, or that there are probably, but not guaranteed, large damping assumptions, it will be necessary to have dynamic tests carried out (see next paragraph) so as to validate these assumptions or dampers.

6.3 - Dynamic tests or tests on footbridges

The carrying out of dynamic tests on a new footbridge, or the testing of an existing footbridge, is a major, very expensive, operation, which should only be considered in very particular cases, and which should only be carried out by contractors who are skilled in this field.
Such an operation should be considered for a new footbridge when the design work has not succeeded in showing that the construction fully satisfies the previous criteria, when the design has imposed the use of dampers that have to be validated, or when the dynamic behaviour is assured using damping assumptions that are greater than those recommended in this guide, but probably required. If the dynamic behaviour can be assured without dampers, and in accordance with the recommendations in this guide, tests will not necessarily have to be carried out.
In the case where dampers are specified right from the start, it is strongly recommended to measure the frequencies and damping of the modes of the completed footbridge before the final sizing of the dampers, in order to adjust such sizing if applicable. The construction programme will have to make allowance for this.
On an existing footbridge, a series of tests can be programmed if the construction has had a number of vibration problems on occasions during its existence, and an improvement is needed.

The following paragraphs give recommendations for the successful completion of such a series of tests. Depending on the size of the construction, the extent of the phenomena encountered and also on what is required to be measured, these recommendations can be adapted.

As far as equipment is concerned, the following systems shall be provided:

- installation of dynamic sensors, (accelerometers or motion measurers), together with a data acquisition unit and a recording system. There must be a sufficient number of accelerometers, positioned at the sags of the theoretical modes.
- installation of a visual monitoring system of the footbridge so as to be able to connect the dynamic measurements to the image of the footbridge. This system will have to be synchronised with the data acquisition unit.
- *Possible* use of a mechanical exciter (of the out-of-balance type for example) so as to be able to characterise precisely the natural vibration modes, natural frequencies, modal masses and modal damping of the footbridge. This system must allow reliable and controlled, horizontal and/or vertical excitation (depending on the footbridge), at programmable frequencies covering the range 0.5 Hz / 3 Hz minimum.

The tests shall proceed in the following way:

6.3.1 - Stage 1: Characterisation of the natural vibration modes

This stage shall be carried out using the dynamic exciter if high accuracy is required of the values measured. In the absence of a dynamic exciter, it is possible to measure the vibrations of the footbridge, in normal service, and, using suitable computer tools, to infer from them the natural frequencies and also the natural vibratory shapes. In this case, certain modes may be forgotten. In addition, it is possible to measure the damping using the logarithmic decrement method, but not the generalised masses, unless a dynamic exciter is used. Dynamic analysis remains possible, but less accurate.

6.3.2 - Stage 2: Crowd tests.

Crowd tests should be programmed, as they enable the dynamic behaviour and the validity of the dampers to be validated. The number of pedestrians is to be determined according to the size of the footbridge, its theoretical characteristics, and also to the complexity of the management of such a crowd. Several dozen pedestrians will be necessary for a representative test.

The tests may include the following elements:

- Random walking, in circles on the footbridge, or continuously, or even from one end to the other.
- Marching in step, at the natural frequencies of the footbridge.
- Marching in step, coming to a sudden halt, in order to measure the damping of the footbridge.
- Running, jumping, kneeling, in order to test the footbridge under extreme stresses.

The tests shall be prepared so as to define the apparent alarm levels, beyond which the tests must be stopped.

If there are dampers, the tests shall be carried out with the dampers working and then, as a second stage, locked into position so as to determine their actual effectiveness.

The tests will have been passed if the vibrations felt during the passage of a desynchronised crowd result in vibrations acceptable to the Owner, tolerable in the case of a synchronised crowd, and not too intolerable in the case of exceptional loading tests (jumping, running, kneeling, vandalism, etc.). In all cases, the decision will lie with the Owner, who must judge the comfort level of his footbridge according to his requirements.

APPENDIX

Appendix 1: Reminders of the dynamics of constructions

Appendix 2: Modelling of the pedestrian load

Appendix 3: Damping systems

Appendix 4: Examples of footbridges

Appendix 5: Examples of the structural design of footbridges

Appendix 6: Bibliography

1. Appendix 1: Reminders of the dynamics of constructions

1.1 - A simple oscillator

1.1.1 - Introduction

A simple oscillator is the basic element of the mechanics of vibrations. Its study enables the characteristic phenomena of dynamic analysis to be understood. In addition, it can be seen that the dynamic analysis of constructions reduces them to simple oscillators.
A simple oscillator comprises a mass *m*, connected to a fixed point via a linear spring of stiffness *k*, and a linear viscosity damper c (figure 1.1).
This oscillator is fully determined when its position *x(t)* is known; a single parameter being enough to describe it, it is said to have 1 degree of freedom (1 DOF): it is also called an oscillator at 1 DOF. The mass may be subjected to a dynamic excitation *F(t)*.
We will still assume that the oscillator carries out small movements around its balance configuration: *x(t)* designates its position around this balance configuration.
An exhaustive study is carried out on this oscillator:

- free damped / non-damped oscillation,
- forced damped / non-damped oscillation.

The "non-damped" case is that with a damping value of zero. It is then said that the system is conservative.
The "free" case is that of zero excitation.
The resonance phenomenon can also be shown: the natural pulsation of a system will be defined. Finally, a brief description of damping will be given.

1.1.2 - Formation of the equations

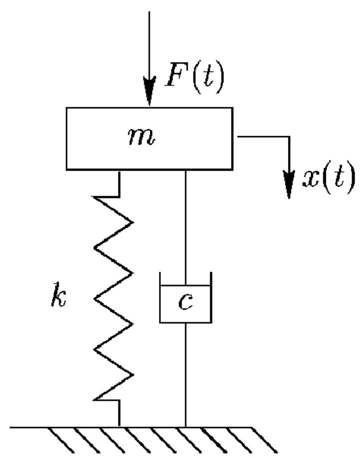


Figure 1.1: Simple oscillator

The mass m (figure 1.1) is subjected to

- the return force of the spring $-k$ x: if the stiffness k is negative, the spring will travel in the direction of movement: this will lead to buckling,

- the viscous dissipation force $-c\,\dot{x}$: if the viscosity c is negative, the construction is unable to dissipate energy: this will lead to dynamic instability,
- the external force *F(t)*.

The fundamental relationship of the dynamics is therefore written:

$$m\ \ddot{x}\ = F - c\,\dot{x} - k\ x \qquad \text{(Eq. 1.1)}$$

which is rewritten in the form:

$$m\ \ddot{x}\ + c\ \dot{x}\ + k\ \ x\ = F \qquad \text{(Eq. 1.2)}$$

In order to solve such an equation, the initial conditions must, obviously, be available:

$$\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= \dot{x}_0 \end{aligned}$$

There is, therefore, a differential equation with constant coefficients and one entry, the force *F(t)*. From a practical point of view, the system being linear, the study of this oscillator can be separated into two rates:

- the natural or free rate: the system is excited only by non-zero initial conditions: the free vibrations of the system are then obtained,
- the forced rate: only the force *F(t)* is non-zero; this force may be sinusoidal, periodic, random or commonplace,

It is typical and practical to divide the equation (1.2) by *m*. We thus get:

- $\omega_0 = \sqrt{\dfrac{k}{m}} = 2\pi\, f_0$: natural pulsation of the oscillator (rad/s); f_0 being the natural frequency of the oscillator (in Hz).
- $\xi = \dfrac{c}{2\sqrt{k\,m}}$: critical damping ratio (without dimension).

The equation (1.2) becomes:

$$\ddot{x}\ + 2\,\xi\,\omega_0\dot{x}\ + \omega_0^2\ \ x\ = \frac{F}{m} \qquad \text{(Eq. 1.3)}$$

1.1.3 - Free oscillation

The system is set to vibrate only by non-zero initial conditions:

$$\ddot{x}\ + 2\xi\,\omega_0\dot{x}\ + \omega_0\ \ x\ = 0$$

$$\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= \dot{x}_0 \end{aligned}$$

The solution of this differential equation with constant coefficients is of the type:

$$x(t) = A\ e^{(rt)}$$

with *r* checking the characteristic equation:

$$r^2 + 2\ \xi\,\omega_0\ r + \ \omega_0^2 \ = \ 0$$

in which the reduced discriminant equals:

$$\Delta \ = \ \omega_0^2\ \ (\xi^2 - 1)$$

It is assumed that ξ is positive (dissipative system) or zero. Thus, different rates are obtained depending on whether ξ is less than, equal to, or greater than 1 (figure 1.2). We will study here only the case that is typical in practice: ξ is strictly smaller than 1.

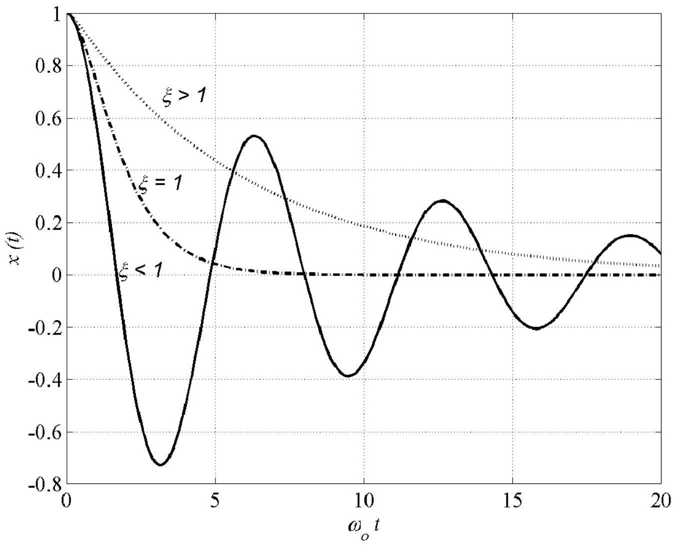


Figure 1.2: Free oscillations

1.1.3.1 Damping $\xi = 0$: the system is non-damped (or conservative)

The solution is therefore:

$$x(t) = A \sin(\omega_0 t) + B \cos(\omega_0 t)$$

A and B are two constants to be determined with the help of the initial conditions:

$$x(0) = x_0 = B$$

$$\dot{x}(0) = \dot{x}_0 = A\omega_0$$

It should be noted that the solution can also be written in the form:

$$x(t) = A \cos(\omega_0 t + \phi)$$

The integration constants are now A and the phase ϕ, which are also obtained with the help of the initial conditions.

1.1.3.2 Damping ξ is strictly between 0 and 1

This is the most interesting case in practice. The solutions of the characteristic equation are:

$$r_{1,2} = -\xi \omega_0 \pm i \omega_0 \sqrt{1-\xi^2}$$

The general solution of the movement equation is written according to one of the three following forms:

$$\begin{aligned} x(t) &= A_1 e^{(-\xi\omega_0 + i\omega_0\sqrt{1-\xi^2})t} + A_2 e^{(-\xi\omega_0 - i\omega_0\sqrt{1-\xi^2})t} \\ &= B_1 e^{-\xi\omega_0 t} \cos(\omega_0 \sqrt{1-\xi^2}\, t) + B_2 e^{-\xi\omega_0 t} \sin(\omega_0 \sqrt{1-\xi^2}\, t) \\ &= C_1 e^{-\xi\omega_0 t} \cos(\omega_0 \sqrt{1-\xi^2}\, t + \phi) \end{aligned}$$

The pulsation ω_a is known as the natural pulsation of the damped system or pseudo-pulsation:

$$\omega_a = \omega_0 \sqrt{1-\xi^2}$$

The constants are obviously determined thanks to the initial conditions.

Comment: It should be noted that, if the damping ξ was negative, there would be a solution that would become divergent: the oscillations would have an amplitude that would increase exponentially. This problem is encountered in wind-excited constructions: the wind can induce a force proportional to its speed and in the same direction; it is then possible to obtain negative damping, as soon as this proportionality factor becomes greater than the damping of the construction.

1.1.4 - Forced vibration

1.1.4.1 Description of the problem

The objective is to resolve the equation (1.3). Various cases can be examined for $F(t)$: F may be harmonic, periodic, random or commonplace.
From a mathematical point of view, the general solution of (1.3) is the sum of the general solution of the equation without a second member (transitory movement or rate) and of a particular solution of the full equation (permanent movement or rate) $x_{SP}(t)$. It can therefore be written in the following form:

$$x(t) = A\,e^{-\xi\omega_0 t}\cos(\omega_a t - \phi) + x_{SP}(t) \qquad \text{(Eq. 1.4)}$$

The constants are determined with the initial conditions.
The transitory rate is associated with the free movement: its influence quickly becomes negligible (in fact, by the end of a few normal periods it has already been cancelled out). During the study with harmonic or periodic excitation, it is not taken into account.

With the help of the harmonic excitation, the notion of transfer function will be introduced. This notion is fundamental to the mechanics of the vibrations.

1.1.4.2 Harmonic excitation

A harmonic excitation is a sinusoidal function:

$$F(t) = F_0 \cos(\omega t) \;=\; \Re(F_0\, e^{i\,\omega t})$$

A particularly interesting solution is in the form:

$$x(t) = A\, cos(\omega_0\, t + \phi)$$

In order to carry out the calculations more simply and more quickly, complex numbers are used. A solution is therefore sought in the form:

$$\tilde{x}(t) = X(\omega)\, e^{i(\omega t - \phi)} \qquad \text{(Eq. 1.5)}$$

with $X(\omega) \in R^+$.
The solution sought is therefore the actual part of $\tilde{x}(t)$: $x(t) = \Re(\tilde{x}(t))$.
(1.5) is injected into the equation (1.3). After simplification by the factor $e^{i\,\omega t}$, we get:

$$e^{-i\phi}\left(-\omega^2 X(\omega) + 2i\xi\omega_0\,\omega X(\omega) + \omega_0^2 X(\omega)\right) = \frac{F_0}{m}$$

The transfer function is then defined, also known as the frequency response, $H_{x,F}(\omega)$, between the excitation and the response (here in movement) by the ratio:

$$H_{x,F}(\omega) = \frac{X(\omega)\, e^{-i\phi}}{F_0\,/\,m} = \frac{1}{-\omega^2 + 2i\xi\omega_0\,\omega + \omega_0^2} \qquad \text{(Eq. 1.6)}$$

This function characterises the dynamics of the system under study. If $\Omega = \omega/\omega_0$ is placed, the dynamic amplification is defined by:

$$A(\Omega) = \omega_0^2 H_{x,F}(\Omega) = \frac{X(\Omega)e^{-i\phi}}{F_0\,/\,k} = \frac{X(\Omega)e^{-i\phi}}{x_{static}} = \frac{1}{1-\Omega^2+2i\xi\Omega} \qquad \text{(Eq. 1.7)}$$

x_{static} is the value of the displacement that would be obtained if a static force F_0 was exerted on the oscillator, with a rigidity of k: it is important to note that this value is taken by $X(\omega)$ at zero frequency. Thus the intercept ordinate of the function $H_{x,F}$ is determined with a static calculation: this value is often called static flexibility. This function $A(\Omega)$ shows that, at a given F_0, the maximum response of the oscillator will depend effectively on the frequency (figure 1.3): for certain frequencies, this response is greater (or even much greater) than the static response.

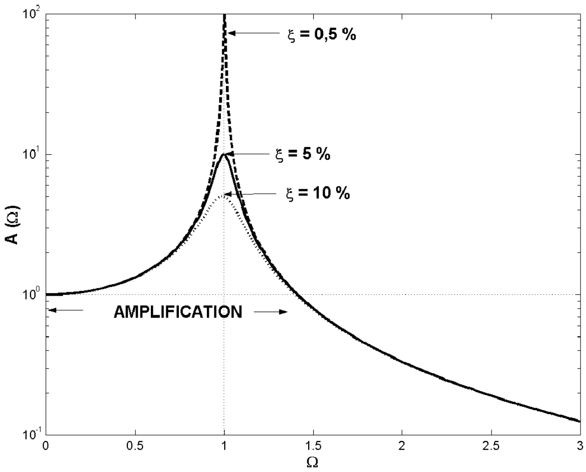


Figure 1.3: resonance phenomenon

The complex solution is therefore:

$$\tilde{x}(t) = \frac{F_0}{m} \frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\xi\omega_0\omega}$$

A study of $A(\Omega)$ shows that, if ξ is between 0 and $1/\sqrt{2}$, the resonance phenomenon is obtained (figure 1.3): $|A(\Omega)|$ allows a maximum for $\Omega = \Omega_R = \sqrt{1 - 2\xi^2}$ which equals:

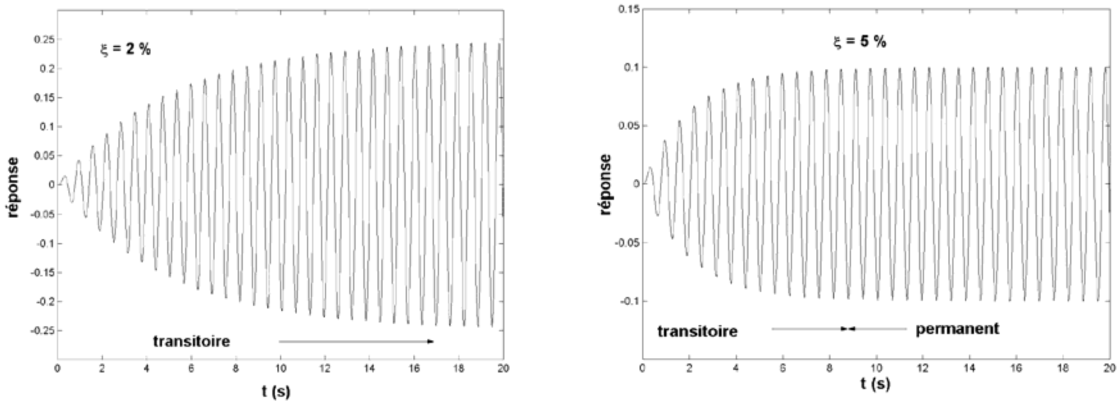
$$\left|A(\Omega_R)\right| = \frac{1}{2\xi}\frac{1}{\sqrt{1-\xi^2}}$$

It can be seen that, the lower the damping (i.e. the more ξ tends towards 0), the higher the amplification to the level of resonance, which tends towards $\frac{1}{2\xi}$: thus, for $\xi = 0.5$ %, it can be seen from figure 1.3 an amplification in the order of 100.

Comment 1: Using this oscillator at 1 DOF, illustrates immediately the difference between "static" and "dynamic". In static, only the amplitude of the excitation affects the amplitude of the response; in dynamic, frequency also has to be taken into account: exciting a construction to its resonance may produce (especially if the construction does not have much damping) major displacements, and thus a high level of stresses in the construction. This is why it is so important to determine the resonance frequencies of a construction as soon as there is dynamic excitation.

Comment 2: It can be seen that resonance is reached at a pulsation value that is less than the natural pulsation of the system ω_0. However, for systems without much damping, the displacement resonance pulsation is combined with the natural pulsation of the system. As far as the phase is concerned, resonance is reached when it equals $\pi/2$.

Comment 3: in the above, no allowance has been made for initial or transitory conditions, as, as has been commented previously, this component of the solution quickly becomes negligible, as can be seen in figure 1.4. However, it is clear that, the higher the damping, the quicker the transitory component becomes negligible.



response transitory response transitory permanent

Figure 1.4: transitory and damping

1.1.4.3 Periodic excitation of period *T*

We return to the previous case. The load function can be developed as a Fourier series:

$$F(t)=\sum_{p=0}^{+\infty} C_n \exp(i\frac{2\pi}{T}t)$$

Thus, by determining the solution $x_p(t)$ for each harmonic p (previous paragraph), we get the solution by superposing (summing) the $x_p(t)$.

1.1.4.4 Commonplace excitation

Taking the Fourier transform of the equation (1.3), we get the frequential equation:

$$X(\omega)=\frac{F(\omega)/m}{\omega_0^2-\omega^2+2i\,\xi\,\omega\omega_0}=H_{x,F}(\omega)\times\frac{F(\omega)}{m}$$

in which $X(\omega)$ and $F(\omega)$ are the Fourier transforms of $x(t)$ and of $F(t)$. By inverse Fourier transform, we then infer x(t), which is expressed using Duhamel's integral:

$$x(t)=\frac{1}{m\,\omega_0\sqrt{1-\xi^2}}\int_0^t F(\tau)\,e^{-\xi\omega(t-\tau)}\sin(\omega_0\sqrt{1-\xi^2}\,(t-\tau))\,d\tau \qquad \text{(Eq. 1.8)}$$

In practice, we carry out a digital integration of the differential equation of the movement (1.3), using a computer program.

1.1.4.5 Random excitation

We assume that the excitation F is modelised by a stochastic process. The response x of the system at 1 DOF is also a stochastic process: it can, therefore, only be understood by the use of values characterising this process (average, standard deviation, autocorrelation function, spectral power density, etc.).
However, the study of random vibrations exceeds the objective of this annexe, and the reader is referred to the reference works.

1.1.4.6 Base excitation

A movement *u(t)* is now imposed on the support: $F(t)$ and the initial conditions are therefore assumed to be zero.
NB: $x(t)$ and $u(t)$ designate displacements within an absolute reference.
We are only interested in the harmonic movement of the support:

$$u(t) = U e^{i \omega t}$$

We are looking once again for a solution in the form: $X(\omega)\, e^{i(\omega t - \phi)}$. By injecting these two expressions into the equation (1.3):

$$\ddot{x} + 2\xi \omega_0 \dot{x} + \omega_0^2 x = 2\xi \omega_0 \dot{u} + \omega_0^2 u$$

We can infer the transfer function between the base displacement and the mass displacement *m*:

$$H_{x,u}(\Omega) = \frac{X(\Omega)\, e^{-i\phi}}{U} = \frac{1 + 2 i \xi \Omega}{1 - \Omega^2 + 2 i \xi \Omega} \qquad \text{(Eq. 1.9)}$$

The modulus of this function is plotted in figure 1.5.

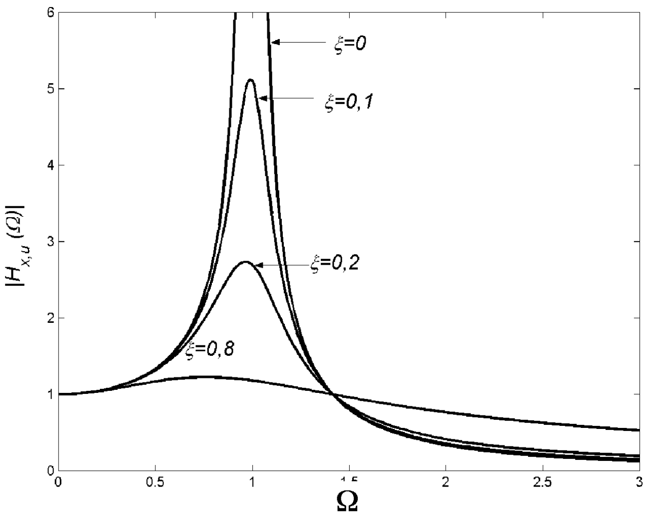


Figure 1.5: FRF in movement, compared with a base excitation

It is interesting to observe the existence of a fixed point:

$$\forall\, \xi,\ |H_{x,u}(\sqrt{2})| = 1$$

Because of this, during a base excitation a compromise must be found in order to optimise the damping: if high ξ limit displacements up to $\sqrt{2}\,\omega_0$, after this value low values of ξ are preferred.

The base excitation is a much more common excitation than could be expected at first glance: indeed, it is seismic excitation (therefore rare) that first comes to mind as a priority to illustrate a base excitation. However, all systems are excited by their support. This is the reason why we try and "isolate" them (elastic bases, Sandow shock cords).
If a system is excited, it transmits a signal (displacement) to its support: this then produces a wave that is transmitted (ground, wall, air, etc.) and causes the displacement of the support of another system.
If the base excitation is periodic or commonplace, an identical study is carried out to that for the forced excitation.

1.1.5 - Damping

1.1.5.1 Generally

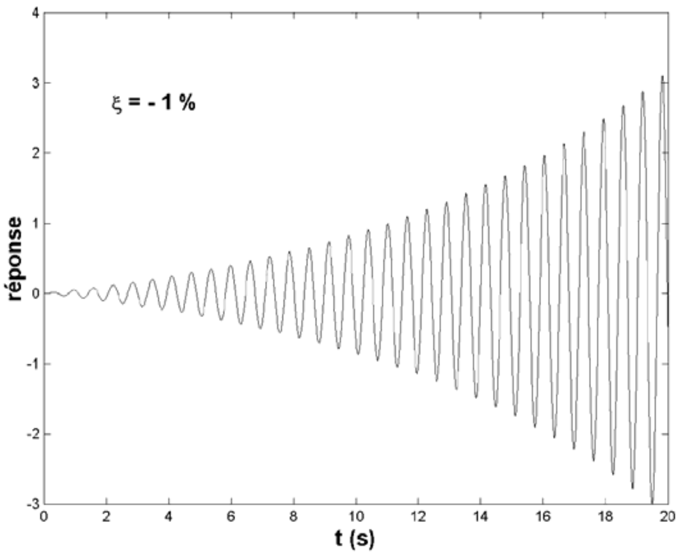
We have seen it: the resonance phenomenon causes displacements in constructions and, thus, stresses (very) much higher than would be obtained by a static calculation. In addition, the maximum amplification is directly linked to the damping. It is therefore essential for this parameter to be estimated properly in order to achieve correct dynamic sizing.
The most common sources of damping are:

- the internal damping linked with the material itself; its value is linked to temperature and the frequency of excitation
- damping by friction (Coulomb): it is linked to the joints between elements of the construction; it is induced by the relative displacement of two parts in contact.

Experimentally, it can be seen that the critical damping ratio does not generally depend on frequency: the damping is then called structural damping:

$$\xi_n = \xi$$

Traditionally, this damping is modelised by viscous damping: this is interpreted as a force that opposes the speed of the construction. It is interesting to know that, sometimes, there exist exciting forces that "agree" with the speed (certain modellings of pedestrian behaviour): this generates "negative damping", which leads to oscillations of the construction that become greater and greater. Instability then exists (see figure 1.6), which can lead to the collapse of the construction.



response
Figure 1.6: Divergence induced by "negative damping"

1.1.5.2 Experimental determination of ξ and ω_0: relaxation tests

When an actual system is studied (ξ is apparently less than 1), modelised by a simple oscillator, it becomes essential to identify its characteristic mechanical parameters, ω_0 and ξ.
This can be done by means of a free vibration test: initial conditions are imposed on the system and its temporal response measured. The following are then measured (see figure 1.7):

- the period T_a between two successive maxima located at t_1 and t_2 ($T_a = t_2 - t_1$): we then get:

$$\omega_0 \sqrt{1-\xi^2} = (2\ \pi)/(T_a),$$

- the value $x(t_1)$ and $x(t_2)$ of two successive maxima: their logarithmic decrement δ can then be inferred:

$$\exp(\delta) = \frac{x(t_1)}{x(t_2)} \cong \exp(\xi\ \omega_0\ T_a) = \exp\left(\frac{2\pi\xi}{\sqrt{1-\xi^2}}\right)$$

I.e.: $\delta = \text{Log}\,(x(t_1) / (x(t_2)) \cong (2\ \pi\ \xi) / \sqrt{1-\xi^2}$

Therefore, if $\xi << 1$ (which is typical), $\xi \cong \delta / (2\ \pi)$

Thus the measurements of T_a and of the logarithmic decrement δ enable the mechanical characteristics of a simple oscillator to be determined.

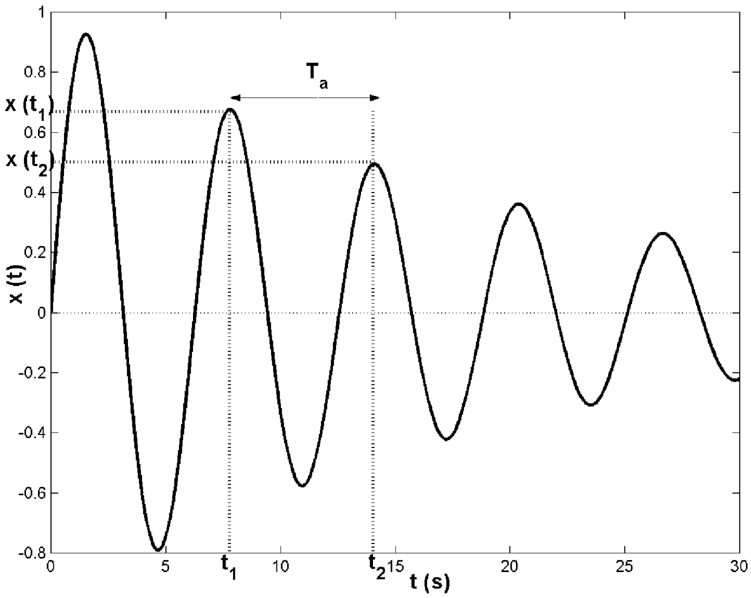


Figure 1.7: response of the 1 DOF oscillator

1.1.5.3 Experimental determination of ω_0 and ξ : peak method

Definition: What is known as the bandwidth $\Delta\Omega$, at -3 *decibels*, is the difference between Ω_1 and Ω_2 such that:

$$\left|A(\Omega_{1,2})\right| = \frac{\left|A(\Omega)\right|_{\max}}{\sqrt{2}}$$

It can be shown that the bandwidth is linked to the damping by the relationship:

$\Delta\Omega = \Omega_2 - \Omega_1 \cong 2\ \xi \sqrt{1-\xi^2} \sim 2\ \xi$ (if the damping is low)

The more this bandwidth tends towards 0, the sharper the resonance and, therefore, the lower the damping (figure 1.8). As a result, to determine Ω_R (and therefore ω_0) and ξ, $|A(\Omega)|$ is plotted experimentally, by carrying out a swept sine test. The pulsation corresponding to the maximum of the graph gives Ω_R. In addition, this graph enables the bandwidth to be obtained and, therefore, ξ.

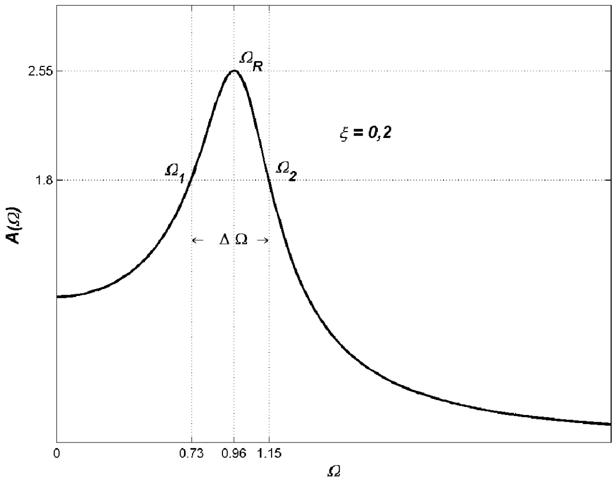


Figure 1.8: Bandwidth for ξ = 0.2

1.1.5.4 Hysteretic damping

The damping used in our model is viscous damping: the damping force is proportional to the speed. This model is very widely used to represent damping. However, in actual fact, there are very few constructions that are subject to this type of damping: its use is due to its simplicity.
During harmonic stresses, this damping model indicates that the energy dissipated in each cycle, at a given amplitude of displacement, depends on the exciting frequency: experimentally, it can be seen that this is not the case. Other models must be determined to overcome this problem. When the stress is harmonic, one simple advantageous model is the hysteretic damping model. This model consists of using a spring unit, the rigidity k_s of which is complex:

$$k_s = k\,(1 + i\,\eta)$$

in which η is known as the structural damping factor. If the dynamic balance of the mass *m* is then written:

$$m\,\ddot{x} + k_s\,x = F(t)$$

Under harmonic excitation, a ξ equivalent to η can be defined in such a way as to conserve the energy dissipated per cycle: $W_d = \pi\,k\,\eta\,x_0^2 = \pi\,\omega\,c\,x_0^2$

Which leads to: $\eta = 2\,\xi\,\dfrac{\omega}{\omega_0}$

1.2 - Linear systems at *n* DOF

Having described the simple oscillator, the discrete systems at *n* DOF can be studied directly, as, from 2 DOF onwards, the methods of understanding vibratory phenomena are the same, whatever the DOF value *n*.

However, studying systems at 2 DOF enables the calculations to be carried out up to the end: that is why, without loss of generality when the calculations are made, they are carried out on systems at 2 DOF.
The general method consists of:

- writing the dynamic equations in a matrix form: the matrices of mass, stiffness and damping are then defined,
- studying the non-damped system: the natural vibration modes and the modal base of the system are then defined,

- if necessary, determining the temporal solution (by going into the modal base, for example).

Thus, in this chapter we will deal with the notions of mass, stiffness and damping matrices, the notions of natural vibration modes of a construction, and of linking between the DOF.

1.2.1 - Formation of the equations

Systems at *n* DOF are formed of n masses connected together by springs and dampers (figure 1.9): if this is not the case, *n* simple oscillators have to be used. Thus, the displacement of a mass is dependant on the displacement of another mass: it is said that the *n* masses are coupled by means of the springs and the dampers.

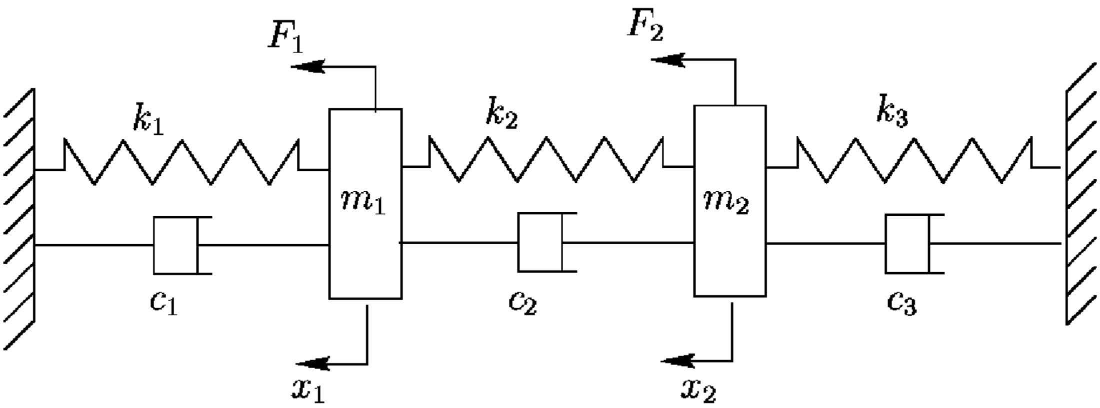


Figure 1.9: System at 2 DOF

Obviously, the *n* DOFs are the positions $x_i(t)_{i=1,\ldots,n}$ of the *n* masses.
For the system represented in figure 1.9, the dynamic equations are written:

$$\begin{cases} M_1\,\ddot{x}_1(t)+(C_1+C_2)\,\dot{x}_1(t)-C_2\,\dot{x}_2(t)+(K_1+K_2)\,x_1(t)-K_2\,x_2(t)=F_1(t) \\ M_2\,\ddot{x}_2(t)+(C_2+C_3)\,\dot{x}_2(t)-C_2\,\dot{x}_1(t)+(K_2+K_3)\,x_2(t)-K_2\,x_1(t)=F_2(t) \end{cases} \qquad \text{(Eq. 1.10)}$$

This system may easily be written in matrix form:

$$\underbrace{\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}}_{[M]}\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix}+\underbrace{\begin{bmatrix} C_1+C_2 & -C_2 \\ -C_2 & C_2+C_3 \end{bmatrix}}_{[C]}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}+\underbrace{\begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2+K_3 \end{bmatrix}}_{[K]}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=\underbrace{\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}}_{[F(t)]}$$

(Eq. 1.11)

Which is therefore written:

$$[M][\ddot{X}(t)]+[C][\dot{X}(t)]+[K][X(t)]=[F(t)] \qquad \text{(Eq. 1.12)}$$

This matrix equation recalls the equation governing the oscillator at 1 DOF. The following names are used:

- [X]: vector of the DOFs,
- [M]: mass matrix,
- [C]: viscous damping matrix,
- [K]: stiffness matrix,
- [F]: vector of external forces.

The mass and stiffness matrices may be found using energy considerations:

- The mass matrix is a matrix associated with the kinetic energy T of the system, which is the sum of the kinetic energies of each mass:

$$T = \tfrac{1}{2} M_1 \, \dot{x}_1^{\,2} + \tfrac{1}{2} M_2 \, \dot{x}_2^{\,2} = \tfrac{1}{2}\, {}^t[\dot{X}]\,[M]\,[\dot{X}]$$

- The stiffness matrix is a matrix associated with the strain energy J of the system, i.e. with the elastic potential energy of each spring:

$$J = \tfrac{1}{2}\; K_1 x_1^2 + \tfrac{1}{2}\; K_2 (x_2 - x_1)^2 + \tfrac{1}{2}\; K_3 x_2^2 = \tfrac{1}{2}\, {}^t[X]\,[K]\,[X]$$

These matrices are, therefore, symmetrical matrices associated with levels of energy.

1.2.2 - Non-damped system

It is assumed here that the damping matrix is zero.

1.2.2.1 Free movement

This particular case ([C] and [F] zero) is fundamental to the study of dynamic systems: it demonstrates the notion of natural vibration modes. The matrix equation (1.11) thus becomes:

$$[M][\ddot{X}] + [K][X] = [0] \qquad \text{(Eq. 1.13)}$$

The initial conditions are therefore not zero, otherwise the system would obviously remain at rest. Solutions are sought in the form:

$[X] = [\phi_0]\, e^{r\,t}$

By injecting into the above equation, the homogenous system at $\phi_{0\,i}$ is obtained, according to:

$$[K + r^2 M]\,[\phi_0] = [0]$$

I.e., for the system at 2 DOF in figure 1.9:

$$\begin{bmatrix} K_1 + K_2 + r^2 M_1 & -K_2 \\ -K_2 & K_2 + K_3 + r^2 M_2 \end{bmatrix} \begin{bmatrix} \varphi_{01} \\ \varphi_{02} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This system permits the zero solution that cannot be satisfactory as soon as there are non-zero initial conditions. To obtain an advantageous solution, the determinant of the system must be zero. We can therefore infer from it the following equation in r:

$$\text{Det}\,([K + r^2 M]) = 0 \qquad \text{(Eq. 1.14)}$$

I.e. for the system at 2 DOF above, we get:

$$((K_1+K_2)+r^2 M_1)\,((K_2+K_3)+r^2 M_2) - K_2^2 = 0$$

For a system at n DOFs this typical equation has 2 n solutions: $\pm\, i\, \omega_{k\,=\,1,\ldots,n}$. With each index k, there is an associated natural vector ϕ_k, solution of the system:

$$[K - \omega_k^2 M]\,[\phi_k] = [0]$$

I.e. for the system at 2 DOFs:

$$\begin{bmatrix} K_1 + K_2 - \omega_k^{\,2} M_1 & -K_2 \\ -K_2 & K_2 + K_3 - \omega_k^{\,2} M_2 \end{bmatrix} \begin{bmatrix} \varphi_{k1} \\ \varphi_{k2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The pulsations ω_k are known as the natural pulsations of the system. The natural vectors [ϕ_k] are the modal vectors of the system. The solution is then written in the form:

$$\left[X(t)\right] = \sum_{k=1,\ldots,n} \left[\phi_k\right] \left(\alpha_k\, e^{-i\omega_k t} + \beta_k\, e^{i\omega_k t} \right)$$

in which the α_k and the β_k are determined by the initial conditions.

The components of the vector ϕ_k are obviously linked. A modal vector is defined to within one multiplicative constant: a standard therefore has to be chosen for these vectors. The most frequent choices are the following:

- one of the components is made equal to 1.
- each mode is unitary: $\|\phi_i\| = 1$,
- a standard is set according to the mass matrix: ${}^t[\phi_i]\ [M][\phi_i] = 1$

If we select the first normalisation: $[\phi_i] = \begin{bmatrix} 1 \\ \phi_{2i} \end{bmatrix}$

It is now possible to express the general solution of the system at 2 DOF, non-damped, under free excitation:

$$\begin{cases} x_1(t) = \alpha_1\, e^{-i\omega_1 t} + \alpha_2\, e^{i\omega_1 t} + \alpha_3\, e^{-i\omega_2 t} + \alpha_4\, e^{i\omega_2 t} \\ x_2(t) = \phi_{21}(\alpha_1\, e^{-i\omega_1 t} + \alpha_2\, e^{i\omega_1 t}) + \phi_{22}(\alpha_3\, e^{-i\omega_2 t} + \alpha_4\, e^{i\omega_2 t}) \end{cases}$$

These expressions are real: x_1 and x_2 can be expressed in trigonometric form:

$$\begin{cases} x_1(t) = a_1 \cos(\omega_1 t) + a_2 \sin(\omega_1 t) + a_3 \cos(\omega_2 t) + a_4 \sin(\omega_2 t) \\ x_2(t) = \phi_{21}(a_1 \cos(\omega_1 t) + a_2 \sin(\omega_1 t)) + \phi_{22}(a_3 \cos(\omega_2 t) + a_4 \sin(\omega_2 t)) \end{cases}$$

The constants are determined using the initial conditions.
In summary, a mode of a linear system is one particular solution of the free, non-damped problem, in such a way that all the components of the vector of the DOFs are synchronous (they reach their maxima and their minima at the same time):

$$\left[X_{\text{mod}e0}\right] = \left[\phi_0\right]\ e^{i\omega_0 t}$$

The natural pulsations ω_i are the square roots of the natural positive values of the matrix $[M]^{-1}\ [K]$. The natural forms of a discrete system are the natural vectors ϕ_i associated with it. The couple $(\omega_i\,,\ \phi_i)$ is called a natural vibration mode.
The modes are sought directly by writing:

- natural pulsation ω_i: this is all of the solutions to the equation:

$$\det([K - \omega_i^2\, M]) = 0$$

- natural vector $[\phi_i]$:

$$[K - \omega_i^2\, M]\ [\phi_i] = [0]$$

1.2.2.2 Orthogonality of the modes

The natural vibration modes are orthogonal in relation to the mass matrix and in relation to the stiffness matrix:

$$\forall\ i, j\ \ {}^t[\phi_i]\ \ [M]\ \ [\phi_j] = m_i\ \delta_{ij}$$
$$\forall\ i, j\ \ {}^t[\phi_i]\ \ [K]\ \ [\phi_j] = k_i\ \delta_{ij}$$

in which δ_{ij} is the Kronecker symbol and m_i and k_i are the mass and the generalised stiffness associated with the mode i: these values will depend on the standard of the natural vectors selected.

1.2.2.3 Harmonic excitation

Once again, before passing to a general forced excitation, we are interested in the harmonic excitation: it is thus that the notion of transfer function is generalised.
It is considered, therefore, that the construction is excited by a force vector of which each component is harmonic and in phase with the other components:

$$[F(t)] = \mathrm{e}^{i\omega t}\ [F_0]$$

It is assumed therefore that the solution is in the form:

$$[X(t)] = \mathrm{e}^{i\omega t}\ [X_0]$$

The matrix equation of the dynamic therefore becomes:

$$([K] - \omega^2\, [M])\ [X_0]\ \mathrm{e}^{i\omega t} = \mathrm{e}^{i\omega t}\ [F_0]$$

From which:

$$[X_0] = ([K]-\omega^2 [M])^{-1} [F_0] = [\alpha(\omega)] [F_0]$$

the matrix $[\alpha]$ is called admittance matrix or dynamic influence factor matrix: this is a response matrix at a frequency between the forced excitation $[F]$ and the response $[X(t)]$.
We can demonstrate the resonance phenomenon: if it is excited at a natural frequency, the response becomes infinite.
The general term of this matrix, α_{ij}, interprets the influence of the component j of the force on the component i of the displacement; if all the components of $[F_0]$ are zero except the component j, we have: $\alpha_{ij}(\omega) = \dfrac{x_{0i}}{F_{0j}}$
This is not forgetting the transfer function of a system at 1 DOF between the excitation $F_{0\,j}$ and the response $x_{0\,i}$. We repeat that α_{ij} is the transfer function between $F_{0\,j}$ and $x_{0\,i}$, or even its frequency response function (FRF).
Comment: There is a resonance peak for each natural frequency of the system. In theory, this peak is infinite. This arises from the (unrealistic) assumption that the system is not damped.

1.2.2.4 Commonplace excitation

Remember the equation to be solved:

$$[M][\ddot{X}(t)] + [K][X(t)] = [F(t)]$$

In order to determine the components of $[X(t)]$, it is always possible to carry out a temporal (digital) integration of the equations. This also requires the inverse of the mass matrix to be determined (high digital cost if the number of DOFs is high). This leads to a differential system of linked equations. If the equations to be solved were not linked, that would simplify their solution: there would be n differential equations of second order to solve (see previous chapter).
That is why it is standard practice to act in a different way: a modal method is used that unlinks the equations.
The principle is to be positioned in modal base: the solution $[X(t)]$ is therefore expressed using a combination of the natural vectors, that form a base. The change of reference is therefore carried out using the transformation:

$$[X(t)] = [\phi][q(t)] = \sum_{i=1}^{n} [\phi_i]\, q_i(t) \qquad \text{(Eq. 1.15)}$$

where:

- $[\phi] = [\phi_1\ \phi_2\ \dots\ \phi_n]$: is a square matrix in which the columns are the natural vectors;
- $[q(t)]$ is the vector of new variables, known as modal variables.

If $[X(t)]$ is replaced by its expression (Eq. 1.15), in the matrix equation of the dynamic, and if this equation is pre-multiplied by ${}^t[\phi]$, we get:

$${}^t[\phi][M][\phi]\,[\ddot{q}(t)] + {}^t[\phi]\,[K]\,[\phi][q(t)] = {}^t[\phi]\,[F(t)]$$

In view of the orthogonality of the modes in respect of the mass and stiffness matrices, we have:

$$\begin{bmatrix} \ddots & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & \ddots \end{bmatrix} [\ddot{q}(t)] + \begin{bmatrix} \ddots & 0 & 0 \\ 0 & k_i & 0 \\ 0 & 0 & \ddots \end{bmatrix} [q(t)] = {}^t[\phi][F(t)]$$

Thus, as the two matrices ${}^t[\phi][M][\phi]$ and ${}^t[\phi][K][\phi]$ are diagonal, we get a system of unlinked differential equations of the type:

$$\ddot{q}_i(t)+\omega_i^2 q_i(t)=\frac{{}^t[\phi_i][F(t)]}{m_i}=p_i(t) \qquad \text{(Eq. 1.16)}$$

The vector $[p(t)]$ is the vector of the modal contribution factors of the force $[F(t)]$: it is the component per unit of generalised mass of $[F(t)]$ in the modal base.
We thus have to refer to the chapter on the oscillator at 1 DOF: Duhamel's integral allows each $q_i(t)$ to be obtained. Its vector $[X]$ can then be inferred by modal superposition, in accordance with equation (1.15).
Using the principle of modal superposition, the study turns from a system with n DOFs to a study of n simple oscillators.
Particular case: For a harmonic excitation, we have:

$$q_i(t) = q_{i0}\cos(\omega t)$$

with:

$$q_{i0}=\frac{p_{i0}}{\omega_i^2-\omega^2}$$

From which

$$[X]=\sum_{i=1}^{n}[\phi_i]\frac{p_{i0}}{\omega_i^2-\omega^2}\cos(\omega t)$$

This expression clearly demonstrates the resonance phenomenon for each natural pulsation.

1.2.3 - Damped systems

In order to simplify the study, only the viscous damping is considered. The equation of the problem is thus the equation (1.12) already described:

$$[M][\ddot{X}]+[C][\dot{X}]+[K][X] = [F]$$

Here again, it is possible to integrate the equations directly. We will even see that this might be essential to make allowance for dissipation effects.
However, we will apply the modal method to our system: we will then have to discuss its validity.
In the following, the couple $(\omega_i, [\phi_i])$ designates the mode i of the equivalent non-damped system. We will break down the equation (1.12) on the basis of natural vectors, following the same route as in the previous paragraph:

$${}^t[\phi][M][\phi]\,[\ddot{q}(t)]+{}^t[\phi][C][\phi]\,[\dot{q}(t)]+{}^t[\phi]\,[K]\,[\phi][q(t)]={}^t[\phi]\,[F(t)]$$

Although the matrices ${}^t[\phi][M][\phi]$ and ${}^t[\phi][K][\phi]$ are, obviously, still diagonal, the same does not necessarily apply to the matrix ${}^t[\phi][C][\phi]$; the modal vectors would appear not to be orthogonal in respect of the damping matrix:

$$\gamma_{ij} = {}^t[\phi_i][C][\phi_j] \neq \text{constant}\times\delta_{ij}$$

Thus, the projection on the basis of natural vibration modes does not unlink the equations from the damped system: the modal method seems to lose a lot of its attraction for determining a digital solution of $[X]$.

1.2.3.1 Modal damping assumption

If the linking due to damping was zero, the modal method would become more attractive. This is only realistic if the distribution of the damping is similar to that of the mass and of the rigidity: such an assumption does not appear to be justifiable.

In practice, very little is known about the distribution of the damping. However, if the construction is not very dissipative and the natural frequencies are clearly separated, it can be shown that the diagonal damping assumption is reasonable: this is known as the modal damping assumption.
Caughey has shown that the natural vibration modes are orthogonal at any damping matrix that is expressed in the general form:

$$[C]=\sum_{k=1}^{N} a_k\,[M]\left([M]^{-1}[K]\right)^{k-1}$$

In the particular case where N equals 2, proportional damping can be found: the damping matrix is expressed according to a linear combination of $[K]$ and $[M]$; it is then clear that, in this particular case, the natural vibration modes of the system are orthogonal in respect of the damping matrix.
Damping is often low if it is well spread out. In this case, it can also be shown that the natural vibration modes of the non-damped construction are fairly close to the modes of the construction. Once again, therefore, the principle of modal superposition is applied, using the modes of the non-damped construction.
Thus, if we take:

$$\xi_i = \frac{\gamma_i}{2\,\omega_i\, m_i}$$

we find once more a system of uncoupled equations:

$$\ddot{q}_i(t) + 2\,\xi_i\,\omega_i\,\dot{q}_i(t) + \omega_i^2\,q_i(t) = p_i(t) = \frac{{}^t[\phi_i][F]}{m_i} \qquad \text{(Eq. 1.17)}$$

Each represents the equation of the movement of a simple damped oscillator under a forced excitation.
In practice, each critical damping ratio ξ_i must be recalibrated experimentally. In fact it is very rare to find a construction with damping that needs simple modelling.
It is normal to consider that the value of the critical damping ratio does not depend on the mode under consideration: $\xi_i = \xi$. This constant ξ is then fixed by experiment or by the regulations.

1.2.3.2 Harmonic excitation – case of systems with low dissipation

The natural modes are therefore assumed to be orthogonal in respect of the damping matrix.
A resolution is carried out by modal superposition: we are looking for $[X]$, the solution to the problem, using modal variables:

$$[X] = [\phi]\,[q]$$

Each modal variable q_i is then subjected to the equation (1.17). The solution sought and the excitation are, traditionally, in the form:

$$\begin{aligned}[q] &= [q_0]\,e^{i\omega t}\\ [p] &= [p_0]\,e^{i\omega t}\end{aligned} \qquad \text{(Eq. 1.18)}$$

$[p]$ always being the vector of the factors of modal contribution to the force $[F]$:

$$p_i = \frac{{}^t[\phi_i][F]}{{}^t[\phi_i][M][\phi_i]} = \frac{{}^t[\phi_i][F]}{m_i} = p_{0i}\,e^{i\omega t}$$

The solution is therefore:

$$[X_0] = \sum_{i=1}^{n} q_{0i}\,[\phi_i] = \sum_{i=1}^{n} \frac{p_{0i}\,[\phi_i]}{\omega_i^2 - \omega^2 + i\,2\,\xi_i\,\omega_i\,\omega} = \sum_{i=1}^{n} \frac{1}{m_i}\,\frac{[\phi_i]\,{}^t[\phi_i]}{\omega_i^2 - \omega^2 + i\,2\,\xi_i\,\omega_i\,\omega}[F]$$

It can then be recognised that each term of this sum is the response of an oscillator at 1 DOF under harmonic excitation. The response of an oscillator at *n* DOF is therefore expressed according to the response of *n* oscillators at 1 DOF**.**

1.2.3.3 Commonplace excitation

Thanks to the modal superposition method, we are led to the study of a system at a DOF under a commonplace excitation. The solution is then given by Duhamel's integral:

$$q_i(t) = \frac{1}{\omega_i \sqrt{1-\xi_i^2}} \int_0^t p_i(t)\, e^{-\xi_i \omega_i (t-\tau)} \sin(\omega_i \sqrt{1-\xi_i^2}\,(t-\tau))\, d\tau$$

which leads to the determination of the solution, by modal superposition:

$$[X] = \sum_{i=1}^{n} [\phi_i]\, q_i(t)$$

1.2.4 - Modal truncation

A complex construction may be modelised by *n* DOF, where *n* is very large. Also, it is normal to represent the physical DOFs by a linear combination of the first *N* modes (they are assumed to be ranged in increasing order of frequency), where *N* is less, or even much less, than *n*.
Such an operation must be carried out with certain precautions: a reliable criterion must be defined to justify this modal truncation.
This approach will depend on the spectrum of the excitation: if it is restricted to a range of pulsation less than ω_N, it may be justified. However, for more thoroughness and accuracy, it is essential to make allowance for these neglected modes. If it is thought that these modes have an almost-static response, it can be shown that is possible to express the neglected modes according to the modes retained**:**

$$\begin{aligned}[X] &\approx \sum_{i=1}^{N} q_i(t)[\phi_i] + \left(\sum_{i=N+1}^{n} \frac{[\phi_i]\,{}^t[\phi_i]}{m_i\, \omega_i^2} \right)[F] \\ &\approx \sum_{i=1}^{N} q_i(t)[\phi_i] + \left([K]^{-1} - \sum_{i=1}^{N} \frac{[\phi_i]\,{}^t[\phi_i]}{m_i\, \omega_i^2} \right)[F]\end{aligned}$$

Matrices with residual stiffness are known as K_{res} and residual flexibility S_{res} such that:

$$[S_{res}] = [K_{res}]^{-1} = [K]^{-1} - \sum_{i=1}^{N} \frac{[\phi_i]\,{}^t[\phi_i]}{m_i\, \omega_i^2}$$

It is very important to note that this matrix depends only on static characteristics ([*K*]) and on the first *N* modes: the neglected modes are thus taken into account without having to calculate them!
Comment: The above expression comes from the modal decomposition of the inverse of the stiffness matrix:

$$[K]^{-1} = \sum_{i=1}^{n} \frac{[\phi_i]\,{}^t[\phi_i]}{m_i\, \omega_i^2} = \sum_{i=1}^{n} \frac{[\phi_i]\,{}^t[\phi_i]}{k_i} \qquad \text{(Eq. 1.19)}$$

If the system is loaded statically, the modal equations (1.17) become

$$\omega_i^2\, q_i = \frac{{}^t[\phi_i][F]}{m_i}$$

and then the modal response is:

$$[X]=\sum_{i=1}^{n}[\phi_i]q_i=\sum_{i=1}^{n}[\phi_i]\frac{{}^t[\phi_i][F]}{\omega_i^2\,m_i}=\left(\sum_{i=1}^{n}\frac{[\phi_i]\,{}^t[\phi_i]}{\omega_i^2\,m_i}\right)[F]=[K]^{-1}[F]$$

By identification, we find once more equation (1.19).

1.2.5 - Rayleigh-Ritz method

1.2.5.1 Rayleigh method

We propose an energy approach in order to determine an estimate of the first natural frequency of a construction: this is the Rayleigh method.

This method is based on a principle of approximation of the field of displacement of the system being studied: we suggest the shape of the construction, i.e., here, a displacement vector $[\widetilde{X}_0]$:

$$[\widetilde{X}(t)]=[\widetilde{X}_0]d(t)$$

This vector must be kinematically permissible.

The strain and kinematic energies associated with this shape are then calculated. The first natural pulsation is then estimated by the ratio:

$$\omega_i^2=\frac{{}^t[\widetilde{X}_0][K][\widetilde{X}_0]}{{}^t[\widetilde{X}_0][M][\widetilde{X}_0]} \qquad \text{(Eq. 1.20)}$$

If the vector $[\widetilde{X}_0]$ is the first natural vector, then the solution obtained is the first natural pulsation. The closer one gets to the natural vector, the better the estimate.

The whole problem is to select a "good" vector $[\widetilde{X}_0]$, in other words to have an idea of the natural vector. In general, the displacement vector obtained under a static load proportional to the value of the masses is a vector close to the first natural vector.

1.2.5.2 Rayleigh-Ritz method

This is, to a certain extent, a generalisation of the Rayleigh method: it enables not only an estimate of the first *N* natural frequencies to be obtained (often *N* is much lower than *n*), but also a close estimate of the natural vectors. The principle is the same: we propose *N* vectors $\{[\widetilde{X}_{0i}]\}_{i=1\ldots N}$ that are independent, kinematically permissible: the displacement vector [X(t)] is then approached by:

$$[X(t)]\approx[\widetilde{X}(t)]=[\widetilde{X}_{01}\ldots\widetilde{X}_{0N}]\begin{bmatrix}d_1(t)\\ \vdots\\ d_N(t)\end{bmatrix}=\sum_{i=1}^{N}[\widetilde{X}_{0i}]d_i(t)=\underbrace{[\widetilde{X}_0]}_{n\times N}[d(t)]$$

We then get an estimate of the first *N* natural frequencies by solving the following problem:

$$\underbrace{{}^t[\widetilde{X}_0][K][\widetilde{X}_0]}_{N\times N}-\omega_i^2\underbrace{{}^t[\widetilde{X}_0][M][\widetilde{X}_0]}_{N\times N}=0$$

If we suggest:

$$[\widetilde{K}] = {}^t[\widetilde{X}_0][K][\widetilde{X}_0]$$
$$[\widetilde{M}] = {}^t[\widetilde{X}_0][M][\widetilde{X}_0]$$

we therefore come to determine the values and natural vectors associated with the problem:

$$[\tilde{K}]-\omega^2[\tilde{M}]=0$$

This problem is more advantageous to solve than the original problem, to the extent that *N* is (much) less than *n*.

Thus, the $\tilde{\omega}_i$ obtained are approximations of the first *N* natural pulsations ω_i of the system. There is a natural vector that matches each of these natural values:

$$\tilde{\psi}_i = \begin{bmatrix} \tilde{\psi}_{i1} \\ \vdots \\ \tilde{\psi}_{iN} \end{bmatrix}$$

We then get an approximation of the natural vector *i* by:

$$[\tilde{\phi}_i]=[\tilde{X}_{01}\cdots\tilde{X}_{0N}][\tilde{\psi}_i]=[\tilde{X}_0][\tilde{\psi}_i]=\sum_{j=1}^{N}[\tilde{X}_{0j}]\,\tilde{\psi}_{ij}$$

1.2.6 - Conclusion

To represent a system at *n* DOF by *n* simple oscillators: this is the fundamental idea of this chapter and of the mechanics of linear vibrations of discrete systems in general. This is interpreted by the diagram in figure 1.10.

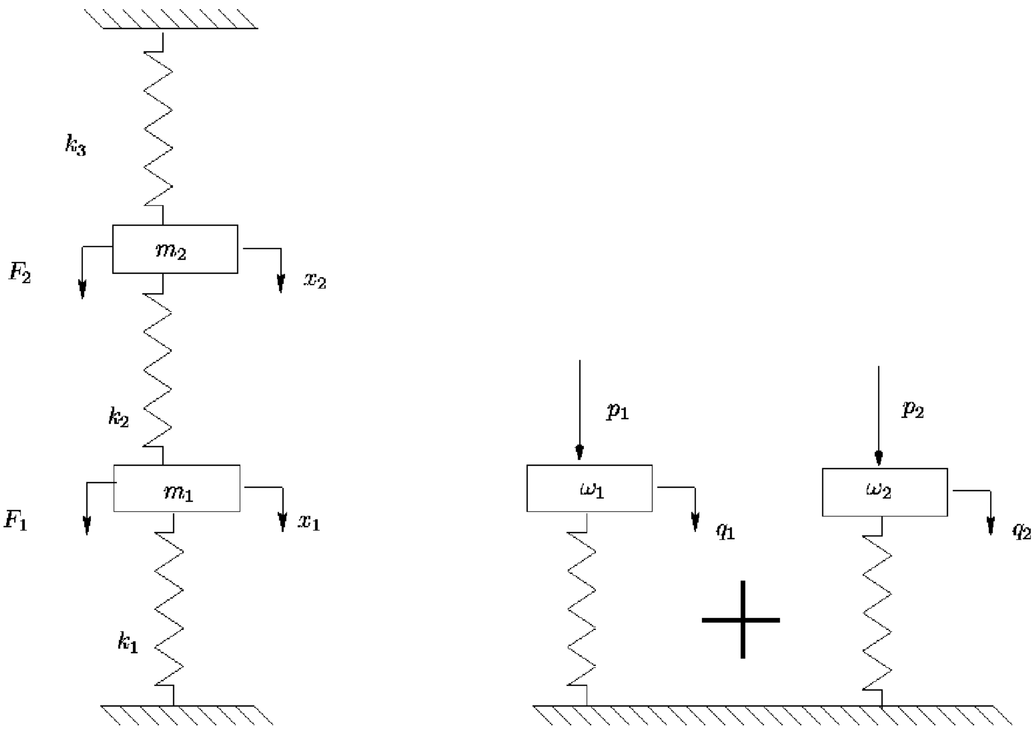


Figure 1.10: Modal approach to a system at 2 DOF

1.3 - Continuous elastic systems

1.3.1 - Generally

Real systems are rarely discrete. Thus, a beam is deformable at any point, has a certain inertia at any point and is capable of dissipating energy at any point.
We are therefore in the presence of a problem of the dynamics of continuous environments: to the statics equations we must therefore add the term that matches the inertia forces.
However, we must not believe that the study of discrete systems was useless: we will see that, from a practical point of view, the study of continuous systems leads finally to the study of discrete systems. Such an approach will necessarily be an approximation, as a discrete system has a finite number of DOFs, whereas a continuous system has an infinite number of DOFs.
The dynamic study of a continuous elastic system requires:

- setting the vibrational problem (formulating the equations):
  - determining the equation for the partial derivatives governing the field of displacement $X\,(x,y,z,t)$: this is the equation of the movement,
  - giving the equations expressing the boundary conditions: these equations may also be equations of partial derivatives,
  - giving the initial conditions.
- solving the problem:
  - calculating the natural modes of the continuous elastic system,
  - calculating the response of the continuous elastic system.

To do this, we assume that the systems studied check the following hypotheses:

- the continuous elastic system is only subject to small strains around its natural state,
- the continuous elastic system has (by definition) a law of linear elastic behaviour; its mechanical characteristics are E, its Young's modulus, and ρ, its density,
- the continuous elastic system vibrates in a vacuum.

1.3.2 - Formation of the equation

In this chapter, we study a beam (to be understood here in its widest sense): this is a rectilinear uni-dimensional continuous environment capable of deforming:

- by tension-compression (bar),
- by pure bending.

We will study these two cases successively.
We will only be studying bi-dimensional constructions with no real loss of generality: the tri-dimensional aspect only adds additional variables.
In a general way, the field of displacement $X(x,y,z,t)$ of a point $M(x,y,z,t)$ on a beam is characterised by its DOFs: $X(x,y,z,t)=\begin{bmatrix} u(x,y,z,t) \\ w(x,y,z,t) \\ \theta(x,y,z,t) \end{bmatrix}$

in which:

- u is the longitudinal displacement, i.e. along the direction of the beam,
- w is the transverse displacement (deflection), i.e. along a direction perpendicular to the beam,
- θ is the rotation around an orthogonal axis in the plane of the construction: this is the slope adopted by the beam, due to the bending moment,

In addition, we are working within a reference linked to the beam: the axis of the beam is combined with (x',x) and the system is in the plane (x,z)

1.3.2.1 Longitudinal movement of a bar

We are only interested therefore in the component $u(x,t)$ of $X(x,t)$. The balance of a length of beam of thickness (assumed infinitesimal) dx and of sectional area S, subjected to:

- the normal force at x,
- the opposite of the normal force at x+dx,
- an external distributed force (pre-stressing) $p(x)$ dx.

leads to the equation of the movement of a bar of constant sectional area:

$$\rho S \frac{\partial^2 u}{\partial t^2} - E S \frac{\partial^2 u}{\partial x^2} = p(x,t) \qquad \text{(Eq. 1.21)}$$

1.3.2.2 Transverse movement

We study a beam in pure bending. We use the Navier hypothesis: straight sections remain straight. We assume, in addition, that transverse shear is negligible (Euler-Bernoulli beam).
We are only interested therefore in the component $w(x,t)$ of $X(x,t)$. The balance of a length of beam of thickness dx, of sectional area S and of inertia I, subjected to:

- the shear force and the bending moment at x,
- the opposite of the shear force and of the bending moment at x+dx,
- an external distributed force $F(x,t)$.

leads to the equation of the movement of a beam in bending of constant sectional area:

$$\rho S \frac{\partial^2 w}{\partial t^2} + E I \frac{\partial^4 w}{\partial x^4} = F(x,t) \qquad \text{(Eq. 1.22)}$$

1.3.2.3 Boundary equations

To solve movement equations, there have to be initial conditions (temporal) and boundary conditions (spatial).
The boundary conditions are very important data, as they set the resonance frequencies of the construction. They are associated with:

- imposed displacements (geometric condition): housings, supports, hinges, etc.
- imposed forces:
    - ends free (natural condition): normal force, zero shear, zero bending moment, etc.
    - localised masses,
    - localised stiffness.

Example 1:
Fixed beam in bending-tension-compression with x = 0 and free at x = L:

$x = 0$	$x = L$
$u = 0$	$N = 0$
$w = 0$	$T = 0$
$\theta = 0$	$M = 0$

Example 2:
Supported beam in bending-tension-compression with $x = 0$ and supported at $x = L$:

$x = 0$	$x = L$
$N = 0$	$N = 0$
$w = 0$	$w = 0$
$M = 0$	$M = 0$

Example 3:

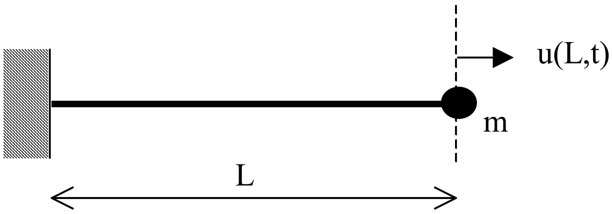


Figure 1.11: Fixed bar (E, S, L) in tension-compression with x = 0 with a localised mass m at x = L.

In order to obtain the boundary condition at $x = L$, the balance of the mass m is written:

$$N(L,t) = m\ddot{u}(L,t)$$

I.e.:

$$-E\,S\,u'(L,t) = m\,\ddot{u}\,(L,t)$$

Example 4:

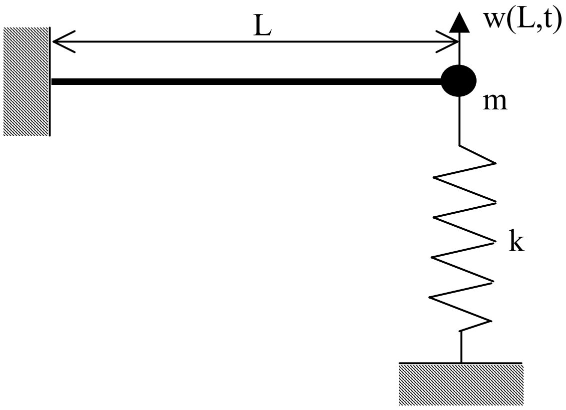


Figure 1.12: Fixed beam (E, I, L) in pure bending with x = 0 with a localised mass m at x = L and a transverse localised stiffness k at x = L.

In order to obtain the boundary condition at $x = L$, the balance of the mass M is written:

$$\begin{aligned} T(L,t) - k\,w(L,t) &= m\,\ddot{w}(L,t) \\ M(L,t) &= 0 \end{aligned}$$

I.e.:

$$\begin{aligned} E\,I\,w^{(3)}(L,t) - k\,w(L,t) &= m\,\ddot{w}(L,t) \\ w^{(2)}(L,t) &= 0 \end{aligned}$$

It can be seen that, in the two examples above, there are six boundary conditions: Two associated with the tension-compression (second order equation) and four associated with the bending (fourth order equation).

1.3.3 - Natural modes of a continuous elastic system

1.3.3.1 Methodology

In order to determine the natural modes of a continuous elastic system, we can use:

- the balance equation,
- the boundary conditions.

One natural mode of a continuous elastic system is a solution $u(x,t)$ to the problem that is written in the form:

$$u(x,t) = \phi(x)\, a(t) \qquad \text{(Eq. 1.23)}$$

In addition, it can be shown that the temporal function is harmonic:

$a(t) = a_0\cos(\omega t + \alpha)$

A mode consists of the data of a natural shape $\phi(x)$ (shape of the construction) and a natural pulsation ω. A mode is thus a stationary wave: all points of the system vibrate in phase.

In order to determine the natural modes of the system, the method is as follows:

- the solution (1.23) is injected into the balance equation without a second member and without damping,
- the time and space variables are separated: there then appears a constant, the sign of which is clear, in order to have a physically acceptable solution,
- the equation (probably differential) is then solved, associated with the spatial value: integration constants appear. The general solution obtained will then permit the natural shapes $\phi(x)$ of the system to be determined,
- the integration constants are determined using boundary conditions: a homogenous system of equations is obtained. As its solution cannot be a zero solution (all the constants zero), in order to determine the constants it will be necessary to impose the nullity of the determinant of the system: this condition gives the characteristic equation of the system,
- the solution of the characteristic equation then provides the natural pulsations of the system.

We apply these principles to the two traditional cases: longitudinal vibration of a bar and bending vibration of a beam.

1.3.3.2 Longitudinal vibrations

The following solution is injected: $u(x,t) = \phi(x)\, a(t)$ in the balance equation of a bar (E, S, L, ρ):

$$\rho S \frac{\partial^2 u}{\partial t^2} - E S \frac{\partial^2 u}{\partial x^2} = 0$$

After simplifications and separation of the variables, we have:

$$\frac{E}{\rho}\frac{\phi''(x)}{\phi(x)} = \frac{\ddot{a}(t)}{a(t)} = \text{constant} = -\omega^2$$

The constant is necessarily negative, otherwise a solution is obtained that grows exponentially over time and is, therefore, physically unacceptable. It is found, therefore, that a(t) is sinusoidal: $a(t) = a_0\cos(\omega t + \alpha)$

In the same way, we get the general form of the natural shapes of a bar:

$$\phi(x) = A\sin\left(\omega\sqrt{\frac{\rho}{E}}x\right) + B\cos\left(\omega\sqrt{\frac{\rho}{E}}x\right)$$

Thus, the modal solution is:

$$u(x,t) = a_0\cos(\omega t + \alpha)\left(A\sin\left(\omega\sqrt{\frac{\rho}{E}}x\right) + B\cos\left(\omega\sqrt{\frac{\rho}{E}}x\right)\right)$$

There therefore remains to be determined the integration and pulsation constants ω:

- a_0 and α are determined with the initial conditions,
- A, B and ω are obtained with the boundary conditions.

Example: Mixed supported bar: the boundary conditions are written:

$$\begin{cases} u(0,t) &= 0 = a(t)\,\phi(0) \\ N(L,t) &= 0 = E\,S\,\dfrac{\partial u}{\partial x}(L,t) = E\,S\,a(t)\,\phi'(L) \end{cases}$$

The following system can be inferred from this:

$$\begin{aligned} B &= 0 \\ \omega\sqrt{\frac{\rho}{E}}\,A\cos\left(\omega\sqrt{\frac{\rho}{E}}L\right) &= 0 \end{aligned}$$

As *B* is zero, in order to obtain a non-zero solution, it must be:

$$\cos\left(\omega\sqrt{\frac{\rho}{E}}L\right) = 0$$

We can therefore infer the natural pulsations from this:

$$\omega_n = \frac{(2n-1)}{2L}\pi\sqrt{\frac{E}{\rho}} \qquad n = 1, 2, \cdots$$

The natural shapes (defined to the nearest multiplicative constant) are therefore:

$$\phi_n(x) = A\sin\left(\frac{2n-1}{2L}\pi\,x\right)$$

In the same way as the discrete case, a generalised stiffness k_n and a generalised mass m_n can be defined:

$$\begin{aligned} m_n &= \int_0^L \rho\,S\,\phi_n^2(x)\,dx \\ k_n &= \int_0^L E\,S\,\phi_n'^2(x)\,dx \end{aligned}$$

We find the relationship: $\omega_n^2 = \dfrac{k_n}{m_n}$

1.3.3.3 Transverse vibrations (pure bending)

The following solution is injected: $w(x,t) = \phi(x)\,a(t)$ into the balance equation:

$$\rho\,S\,\frac{\partial^2 w}{\partial t^2} + E\,I\,\frac{\partial^4 w}{\partial x^4} = 0$$

After simplifications and separation of the variables, we have:

$$\frac{E\,I}{\rho\,S}\,\frac{\phi^{(4)}(x)}{\phi(x)} = -\frac{\ddot{a}(t)}{a(t)} = \text{constante} = \omega^2$$

Here again, the sign of the constant arises from a physically acceptable solution; we find that $a(t)$ is sinusoidal. In addition, the differential equation governing ϕ is of the fourth order:

$$\frac{d^4\phi}{d\,x^4}(x) - \frac{\rho\,S}{E\,I}\,\omega^2\,\phi(x) = 0$$

If we take $k = \sqrt[4]{\dfrac{\rho\,S\,\omega^2}{E\,I}}$, the general solution of the above equation (therefore the shape) is:

$$\phi(x) = A\sin(kx) + B\cos(kx) + C\sinh(kx) + D\cosh(kx)$$

We use the boundary conditions to determine the integration and pulsation constants.
Example: Beam supported at each end:
the boundary conditions are written:

$$\begin{array}{rcl} w(0,t) & = & 0 \\ M(0,t) & = & 0 = E\,I\,w''(0,t) \\ w(L,t) & = & 0 \\ M(L,t) & = & 0 = E\,I\,w''(L,t) \end{array}$$

Whence the system:

$$\begin{array}{rcl} B+D & = & 0 \\ -B+D & = & 0 \\ A\sin(kL)+B\cos(kL)+C\sinh(kL)+D\cosh(kL) & = & 0 \\ -A\,k^2\sin(kL)-B\,k^2\cos(kL)+C\,k^2\sinh(kL)+D\,k^2\cosh(kL) & = & 0 \end{array}$$

As this system is homogenous, there can only be a non-zero solution if the determinant of the system is zero; we can infer from that that characteristic equation which reduces to:
$\sin(kL) = 0$
The resolution of this equation leads to the natural pulsations:

$$\omega_n = \frac{n^2\,\pi^2}{L^2}\sqrt{\frac{E\,I}{\rho\,S}}$$

and, finally, to the mode shapes:

$$\phi_n = A\sin\left(\frac{n\,\pi\,x}{L}\right)$$

A generalised mass and stiffness can also be defined:

$$\begin{array}{rcl} m_n & = & \int_0^L \rho\,S\,\phi_n^2(x)\,dx \\ k_n & = & \int_0^L E\,I\,\phi_n''^2(x)\,dx \end{array}$$

1.4 - Discretisation of the continuous systems

1.4.1 - Introduction

As has been mentioned, few modal analysis problems of continuous elastic systems have an analytical solution. That is why it is necessary to use approximation methods:

- direct mass discretisation method: the construction is approached by masses and springs,
- Rayleigh-Ritz method: the spatial variation of functions is apparently postulated,
- finite element method: the construction is broken down into sub-sections, to which a Rayleigh-Ritz method is applied.

These methods are discretisation methods: we pass from an infinite number of DOFs for continuous systems to a finite number of DOFs.

1.4.2 - Direct discretisation of masses

This method may depend heavily on the aptitude of the "modeliser" to discretise a construction: it is, however, very important, as it is widely used to carry out civil engineering modelisations (particularly for earthquake-protection design). It also permits the mass and stiffness matrices to be obtained very simply.
This method consists of:

- dividing the construction up geometrically,

- concentrating the mass of each divided part on a node (in general its centre of gravity) in the form of:
  - mass acting in translation,
  - inertia of the mass acting in rotation.

As a consequence, the DOFs of the discrete system are the "physical" DOFs associated with the point on which the mass is concentrated: it is supposed, therefore, that each section "strictly follows" the DOF of the node, whence the importance of the choice of dividing up. Such a method leads to a diagonal mass matrix. As far as rigidity is concerned, the stiffness is determined by connecting 2 DOFs together: the rigidity matrix is not necessarily diagonal.
Example: Discretisation of a fixed/free bar (figure 1.13): the first node is located at 1/3 of the length, the second at 2/3 of the length and the last one at the end. Each node is allocated half of the mass of each element to which it is connected; the stiffness of each spring is calculated from the rigidity of a bar element.
The choice of k_i and m_i is important and determines the accuracy that there will be in the natural frequencies. A choice can be made for a division into three sections of the same length:

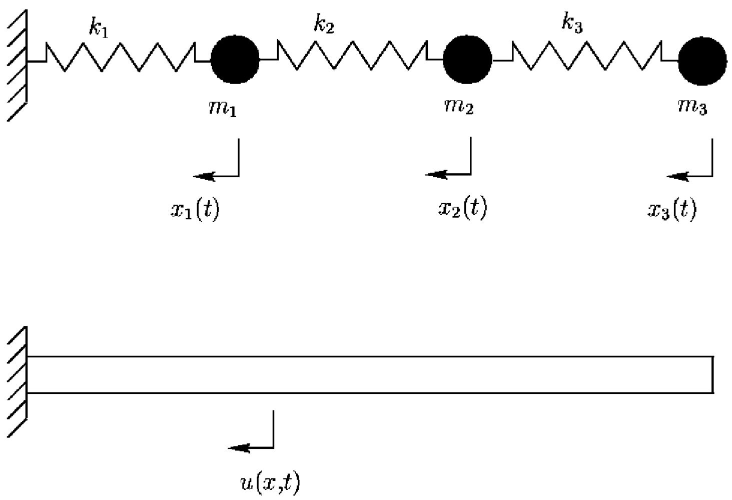


Figure 1.13: Direct discretisation of the mass of a bar

1.4.3 - Rayleigh-Ritz method

As the system to be calculated is continuous, it is determined as soon as the continuous functions describing the DOFs of each point are known: translations, rotations. It is proposed therefore to find out about these continuous functions $u(M, t)$ by the determination of a finite number of parameters $\{a_i\ (t)\}_{i\ =\ 1,\ldots,N}$ (comprising the vector $[a(t)]$) thanks to the approximation functions $\psi_i(M)$:

$$u(M,t) \cong \tilde{u}(M,t) = \sum_{i=1}^{N} \psi_i\ (M)\, a_i\ (t) = {}^t[\psi][a(t)]$$

with:

- $a_i(t)$: generalised coordinate. These are our unknowns to be determined: they are grouped together in the vector of generalised coordinates $[a(t)]$; the dimension n of the vector $[a(t)]$ gives the number of DOFs of the discretised system,
- $\psi_i(M)$: given spatial function. It must be kinematically permissible and **defined** over the whole construction, which occupies the domain V.

Thus, when the generalised coordinates are known, using the approximation functions $\psi_i(M)$, we find u at any point in the system.
The general method consists of:

- Writing the kinetic energy, the system strain energy, and also the work of external forces according to the field approached $\tilde{u}$.
- Re-writing these energies in matrix form, by demonstrating the vectors of the generalised coordinates:

$$T = 1/2\,{}^t[\dot{a}(t)][M][\dot{a}(t)]$$
$$J = 1/2\,{}^t[a(t)][K][a(t)]$$
$$W_{ext} = {}^t[a(t)][F] = {}^t[F][a(t)]$$

- The matrices [M] and [K] which then appear are the matrices of the mass and the stiffness of the discretised system: the governing equation [a(t)] is then:

$$[M][\ddot{a}(t)]+[K][a(t)]=[F(t)]$$

- In order to determine the natural modes (ω, [$\phi(x)$]), we use the procedure described in the chapter on discrete systems with n DOF; we resolve:

$$[K][\phi(x)]=-\omega^2[M][\phi(x)]$$

Let us follow the procedure described in the introduction to this chapter:

- Calculation of the kinetic energy:

$$T=1/2\int_V \rho\dot{u}^2(M,t)\ \mathrm{d}V=1/2\,{}^t[\dot{a}(t)]\underbrace{\left(\int_V \rho[\psi]\,{}^t[\psi]\mathrm{d}V\right)}_{[M]}[\dot{a}(t)]$$

We infer from this the expression of the mass matrix of the discretised system:

$$[M]=\int_V \rho[\psi]\,{}^t[\psi]\mathrm{d}V$$

In the simple case of a bar or a beam, the mass matrix therefore becomes:

$$[M]=\rho S\int_0^L[\psi(x)]\,{}^t[\psi(x)]\mathrm{d}x$$

- Calculation of the strain energy:

The general calculation of the strain energy is not made here: depending on the RDM hypotheses, we get different "simplified" expressions, depending on whether the system is modelised by a beam, a bar, etc. We do recall, however, the most typical energies:

Tension-compression	$1/2\,E\,S\int_0^L(u'(x))^2\ \mathrm{d}x$
Bending	$1/2\,E\,I\int_0^L(w''(x))^2\ \mathrm{d}x$

We can then infer from this the stiffness matrices:

Tension-compression	$[K]=E\,S\int_0^L[\psi'(x)]\,{}^t[\psi'(x)]\mathrm{d}x$
Bending	$[K]=E\,I\int_0^L[\psi''(x)]\,{}^t[\psi''(x)]\mathrm{d}x$

- Calculation of the work:

$$W_{ext}=\int_V {}^t[u(M,t)][f(M,t)]\mathrm{d}V\cong{}^t[a(t)][F(t)]={}^t[F(t)][a(t)]$$

Knowing all these matrices, we can then infer from them the balance equation:

$$[M][\ddot{a}(t)]+[K][a(t)]=[F(t)]$$

The whole arsenal developed for systems at n DOF can then be applied:

- calculation of the natural modes of the discrete system,
- response to an external force, etc.

Once the discrete problem has been solved, we can return immediately to the continuous system. Thus, if the natural vectors [ϕ_i] of the discrete system associated with the natural pulsations ω_i have been determined, the first n modes (close) of the continuous system are determined by the same natural pulsation ω_i and by the mode shape $u_i(x)$:

$$u_i(x) = {}^t[\psi(x)][\phi_i]$$

Comment 1: As many modes are determined as there are DOFs. In practice, it can be estimated that only the first $n/2$ modes are determined with sufficient accuracy.
Comment 2: Accuracy is improved when the number of DOFs is increased. The estimate of the natural pulsations by the Rayleigh-Ritz method is always made by excess.
Comment 3: When the system being studied is formed from beams, it is normal to use polynomes for the functions $\psi_i(x)$. For example for a fixed/free beam, we choose: $\psi_i(x) = x^i$ with $i \geq 2$. It is easy to check that these functions are all kinematically permissible.
Comment 4: When $n = 1$, this method is known as Rayleigh's method.
Comment 5: The static shape of the construction subjected to a load proportional to its self-weight gives, in general, a good approximation of the mode shape of the first mode.

1.4.4 - Finite element method

1.4.4.1 Weakness of the Rayleigh-Ritz method

Although it is easy to determine the functions $\psi_i(x)$ for a straight fixed/free beam, the same does not necessarily apply to systems that are hardly more complex, such as, for example, two beams rigidly fixed at one end but non-collinear: in this case, how do we choose the interpolation functions? The simple solution is to "break" the system: we turn therefore to the finite element method.

1.4.4.2 Brief reminders

The construction is sub-divided into fields of a simple geometric shape (of which we know the analytical expression), the contour of which is supported at points (the nodes): these are the elements.
These elements have a sufficiently simple shape for it to be possible to determine functions that are kinematically permissible on each of these elements: these are generally polynomes.
We then apply the Rayleigh-Ritz method to each of the elements: we calculate the kinetic and strain energies for each element. We can then infer the kinetic and strain energies for the whole construction: we use the additivity of the energy. We can then infer the mass and stiffness matrices of the construction: we turn once again to the study of a system with n DOFs.
NB: The approximation functions, called here interpolation functions, are selected in such a way that the generalised coordinates are the values of the displacements of the nodes: the generalised coordinates thus have a physical meaning that enables a continuous displacement field to be obtained over the construction.

We can also determine a load vector by writing, in matrix form, the work of the external forces applied to the system. We can therefore study the response of a continuous system subjected to any load.

2. Appendix 2: Modelling of the pedestrian load

Pedestrian footbridges are subjected mainly to the loads of the pedestrians walking or running on them. These two types of loading must be treated separately, as there is a difference between them: when walking, there is always one foot in contact with the ground, but the same does not apply when a person starts to run (figure 2.1).

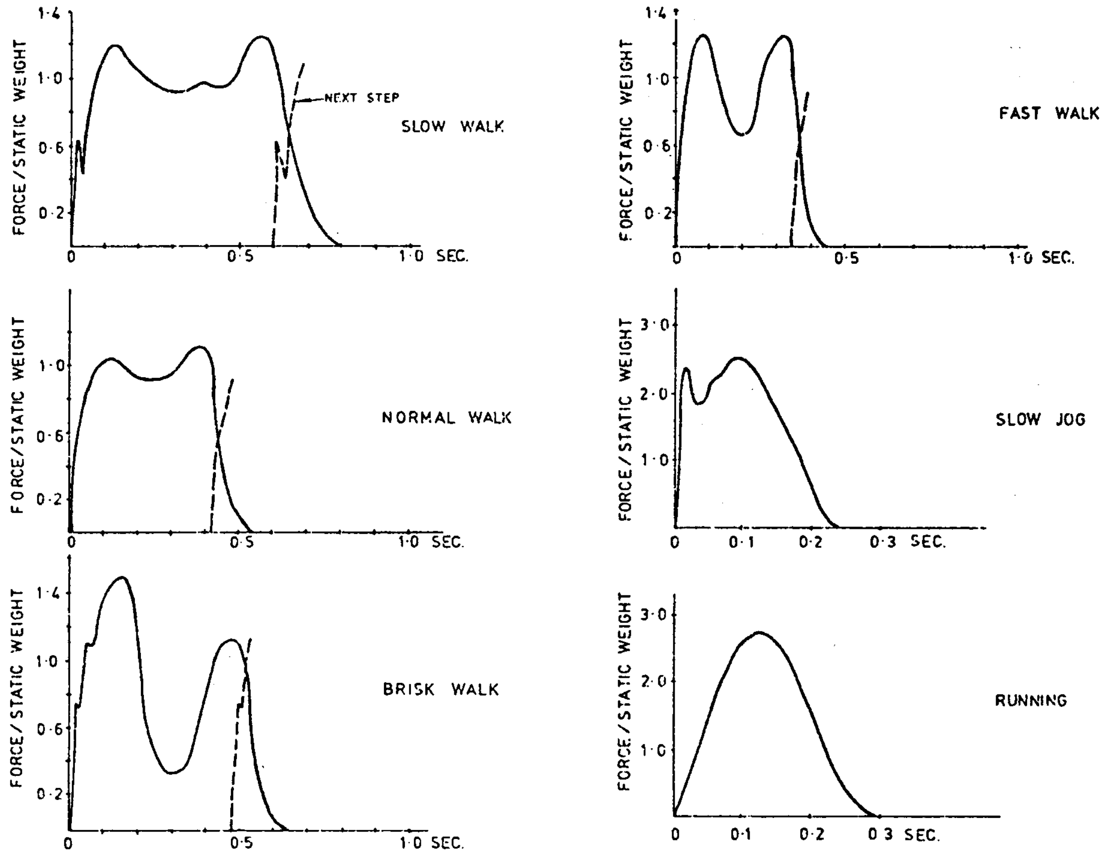


Figure 2.1: changes of the force over time for different types of step (Ref. [17])

Whatever the type of load (walking or running), the force created comprises a vertical component and a horizontal component, it being possible to break down the horizontal component into a longitudinal component (along the axis of the footbridge) and a transverse component (perpendicular to the axis of the footbridge). In addition, apart from the individual action of a pedestrian, we will consider making allowance for a group of people (crowd) and also the stresses connected with exceptional loadings (inaugurations, deliberate excitation, etc.). Finally, when they exist, we will indicate the recommendations specified in the Eurocodes covering dynamic models of pedestrian loading.
For activities without displacement (jumping, swaying, etc.), experimental measurements show that it is preferable to make allowance for at least the first three harmonics to represent the stresses correctly, while, for the vertical component of walking, we can restrict ourselves to the first harmonic, which can be described sufficiently as *F(t)*. The values of the amplitudes and of the phase shifts, arising from experimental measurements, are set out below for the various components of *F* in the case of walking and of running. It should be noted, however, that the values one can find in the literature are approximate, in particular as far as the phase

shifts (φ_i); the values provided are, therefore, orders of magnitude corresponding to an average displacement.
As can be seen in figure 2.1, the shape of the stress changes between walking and running, and also according to the type of walking or running (slow, fast, etc.). The amplitudes G_i are therefore linked to the frequency f_m and the values indicated subsequently are therefore average values, corresponding to "normal" walking or running.

2.1 - Walking

Walking (as opposed to running) is characterised by continuous contact with the ground surface, as the front foot touches the ground before the back foot leaves it.

2.1.1 - Vertical component

In normal walking, the vertical component, for one foot, has a saddle shape, the first maximum corresponding to the impact of the heel on the ground and the second to the thrust of the sole of the foot (see figure 2.2). This shape tends to disappear when the frequency of the walking increases, until it is reduced to a semi-sinusoid when running (see figure 2.1).

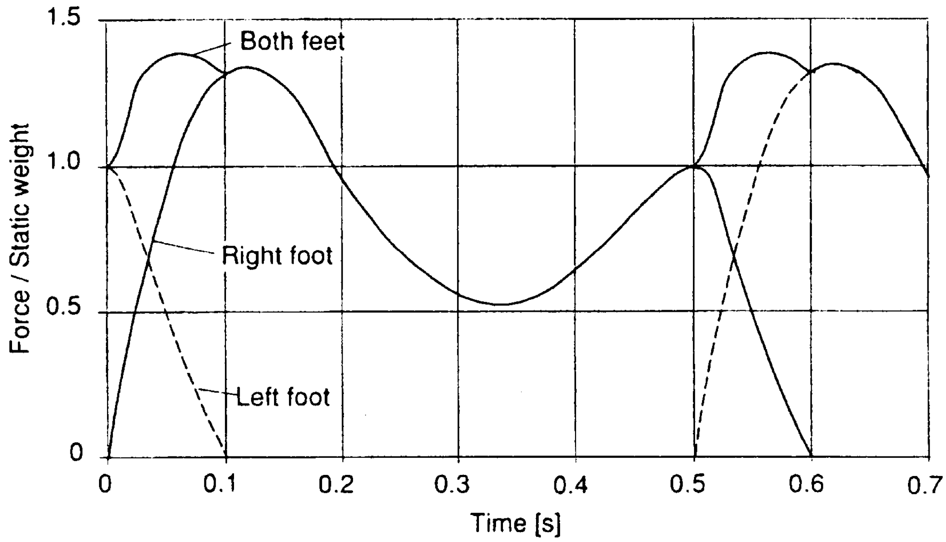


Figure 2.2: force resulting from walking (f_m = 2 Hz) (Ref. [3])

For normal (non-obstructed) walking, the frequency may be described as a Gaussian distribution of average 2 Hz and standard deviation 0,20 Hz approximately (from 0.175 to 0.22 depending on the authors). At the average frequency of 2 Hz (f_m = 2 Hz), the values of the Fourier transform coefficients of $F(t)$ are as follows (as remarked previously, the first three terms are taken into account, in other words n = 3, the factors of the terms of greater order being less than 0.1 G_0):

$G_1 = 0{,}4\ G_0$; $G_2 = G_3 \approx 0{,}1\ G_0$;

$\varphi_2 = \varphi_3 \approx \pi/2$.

As can be seen above, the values of G_i and of φ_i for the terms beyond the first are hardly precise, this being due, on the one hand, to uncertainties when taking measurements and, on the other hand, to differences between one person and another.
Comment: for a frequency of walking of 2.4 Hz, the recommended value of G_1 is 0.5 G_0, the other values being unchanged. In the same way, for slow walking (1 Hz), we have $G_1 = 0{,}1 G_0$.

We have plotted on figure 2.3, for a person of 700 N walking at a frequency f_m of 2 Hz and over 1 second, the change in *F(t)*, taking into account one, two or three harmonics in the previous development. We can see that only when taking the first three harmonics into account can the saddle shape be found. Apart from the shape of the signal, the difference lies in the frequential content of the excitation, a fundamental aspect when calculating response.

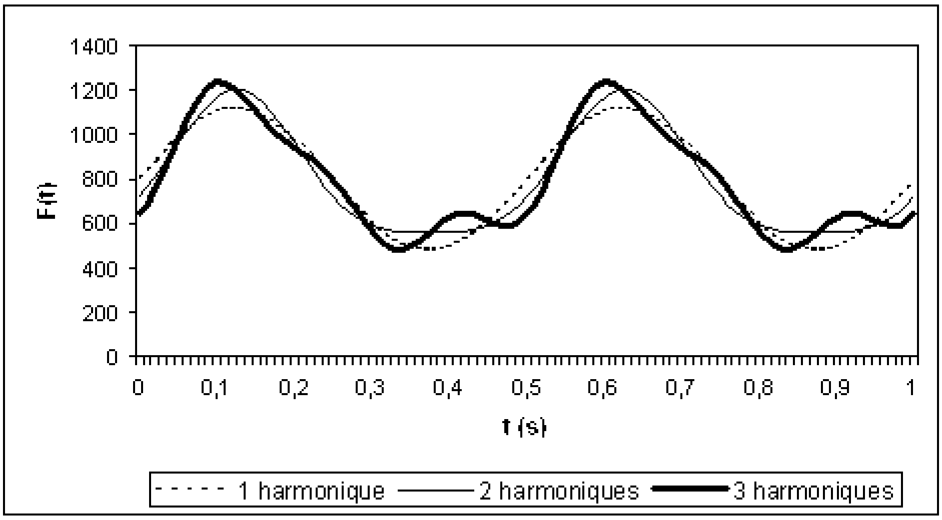


Figure 2.3: walking – vertical component (f_m = 2 Hz)

It should be noted that, in the annex to Eurocode 1 covering dynamic models of pedestrian loads, the recommended load was (DLM1):

$$Q_{pv} = 280\sin(2\pi f_v t)$$

which corresponds, in fact, to $0.4\ G_0$ with G_0 = 700 N, the weight of the pedestrian. This was, therefore, the first term (dynamic) of Fourier's transform. This model has since been deleted from the Eurocode, but its physical reality remains no less relevant.

2.1.2 - Horizontal component

The horizontal component of the load is, admittedly, of less intensity, but cannot, however, be neglected as it can be a source of problems. It is apparent that people are very sensitive to being moved sideways and walking is rapidly disturbed, leading, for example, to keeping close to the handrail, to a feeling of insecurity or even the closing of the footbridge, etc.

As announced previously, we will investigate successively being moved transversely and longitudinally (although longitudinal movement is more rarely a problem on footbridges). The transverse component, corresponding to changing from one foot to the other when walking, occurs, therefore, at a frequency of half that of the frequency of walking (1 Hz for f_m = 2 Hz). On the other hand, the longitudinal component is mainly linked to the frequency of walking (see figure 2.4).

In order to present Fourier's transform of the transverse component (at the frequencies $f_m/2$, f_m, $3f_m/2$) according to the basic frequency of walking f_m, the solution generally used is to modify the presentation in the following form:

$$F(t) = \sum_{i=1/2}^{n} G_i \sin 2\pi i f_m t$$

i therefore being able to have (non-whole) values of 1/2, 1, 3/2, 2, etc. In addition, as the phase shifts are close to 0, they do not therefore appear in the previous expression. Finally, as opposed to the vertical force, the transverse and longitudinal components do not have, of course, a static part (no constant term in the expression of *F(t)*).

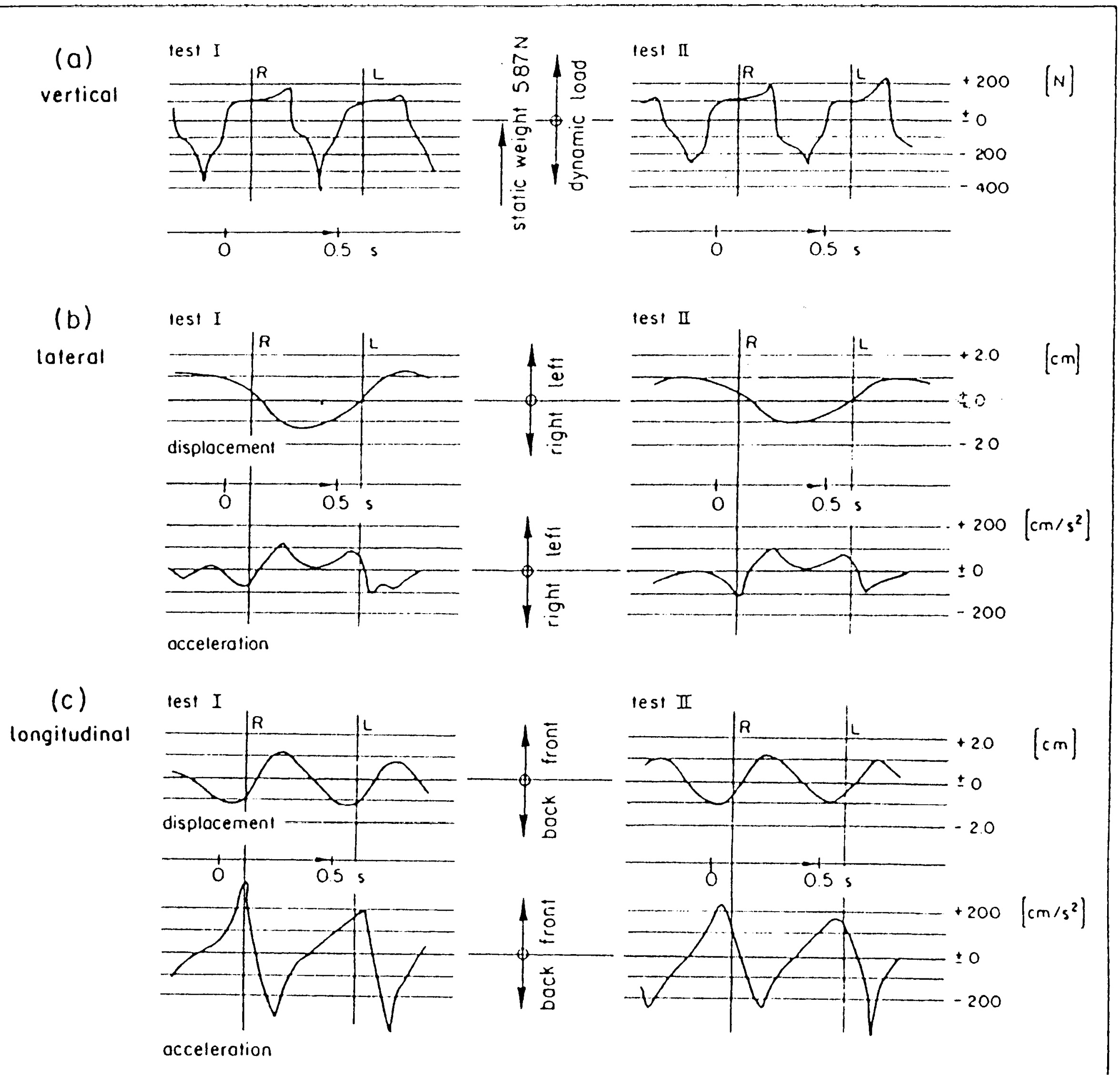


Figure 2.4: changes in the vertical force (a), the transverse (b) and longitudinal (c) components of the displacement (and of the acceleration) measured – walking at 2 Hz Ref. [22]

2.1.2.1 Transverse component

As indicated above, tests have shown that the main amplitudes of this component are located at a frequency of about half that of the vertical component, a frequency corresponding to the lateral oscillations of the centre of gravity of the body when walking. The corresponding values of the Fourier coefficients are thus as follows:

$$G_{1/2} = G_{3/2} \approx 0.05\ G_0\ ;\ G_1 = G_2 \approx 0.01\ G_0.$$

As for the vertical component (G_0 = 700 N, f_m = 2 Hz), we have marked on figure 2.5 the transverse component, taking into account from one to four harmonics. As expected, it can be seen that, on the one hand, the frequency is half that of the vertical component (in other words the period is double) and, on the other hand, that taking into account the first four terms gives an appearance close to those measured in figure 2.4 (b).

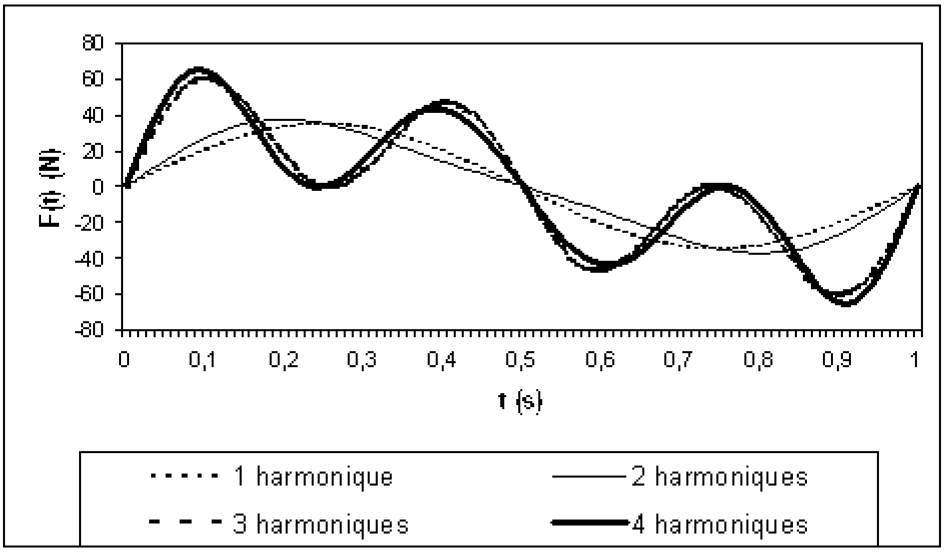


Figure 2.5: walking – transverse component (f_m = 2 Hz)

2.1.2.2 Longitudinal component

The main frequency associated with this component is approximately the same as for the vertical component (f_m = 2 Hz). Its oscillations correspond, for each step, initially to the contact of the foot with the ground, then to the thrust exerted subsequently. For this component, the values of the Fourier coefficients are:

$G_{1/2} \approx 0.04\ G_0$; $G_1 \approx 0.2\ G_0$; $G_{3/2} \approx 0.03\ G_0$; $G_2 \approx 0.1\ G_0$.

We have shown in figure 2.6, under the same conditions as for the other components (G_0 = 700 N, f_m = 2 Hz), the change in this component, depending on the number of harmonics taken into account, the previous comments remaining valid.

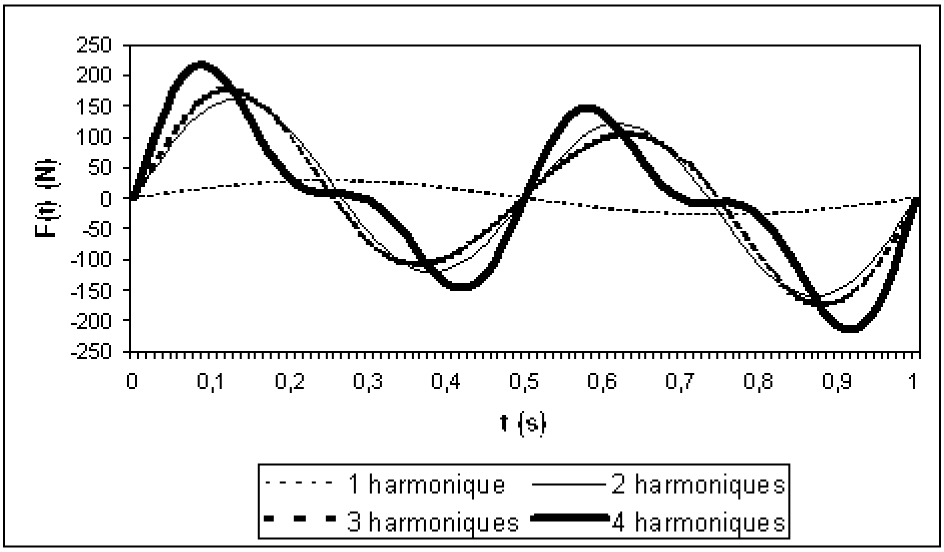


Figure 2.6: walking – longitudinal component (f_m = 2 Hz)

It should be noted that, in practice, the longitudinal component of the walking force of a pedestrian has, in general, little influence on most footbridges. The influence is greater when the footbridge bears on overhanging, flexible piers, for which the longitudinal component of the pedestrian leads to the bending of the pier.

2.2 - Running

As indicated in the introduction, running is characterised by a discontinuous contact with the ground (see figure 2.7 (a)), the frequency f_m generally being between 2 and 3.5 Hz.

2.2.1 - Vertical component

In an initial approximation, the vertical component of the load can be approached by a simple sequence of semi-sinusoids, represented using the following expression:

$$F(t)=k_p G_0 \sin(\pi t/t_p) \; for \; (j-1)T_m \le t \le (j-1)T_m + t_p$$

$$F(t)=0 \; for \; (j-1)T_m + t_p < t \le jT_m$$

with:
- k_p : the impact factor ($k_p = F_{max} / G_0$),
- j : the step number (j = 1, 2, etc.),
- F_{max} : the maximum load,
- G_0 : the weight of the pedestrian,
- t_p : the period of the contact,
- T_m : the period (T_m = $1/f_m$, noted T_p on fig.3.4(b)), f_m being the frequency of running.

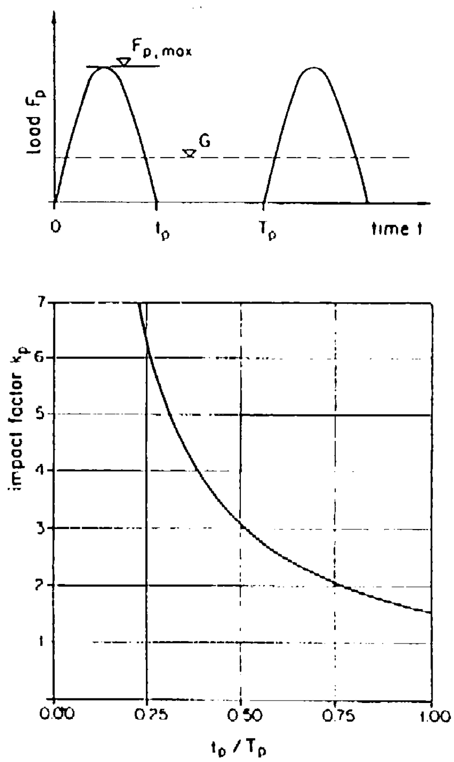


Figure 2.7: running – change in the force according to time (a); impact factor according to the relative period of contact (b) Ref [3]

The period of contact t_p in this model is simply the half-period T_m, which enables the expression of F(t) to be written also in the following form:

$$F(t)=k_p G_0 \sin(2\pi f_m t) \; for \; (j-1)T_m < t \le \left(j-\tfrac{1}{2}\right)T_m$$

$$F(t)=0 \; for \; \left(j-\tfrac{1}{2}\right)T_m < t \le jT_m$$

This approximation of the period of contact t_p is an over-estimation of the values measured experimentally, which are represented according to the frequency of running on the graph in figure 2.8. As for the values of the impact factor k_p, these are inferred from figure 2.7 (b), according to the relative period of contact (t_p/T_m). As an example, running at a frequency of 3 Hz (thus a period T_m = 0.33) gives t_p = 0.17 s and k_p = 3 when t_p/T_p = 0.5 (instead of k_p = 2.4 when t_p/T_p = 0.61). As announced, we can therefore see from this example that the semi-sinusoidal approximation is conservative.

It is also possible to use, for running, a Fourier transform, which has the advantage of not revealing explicitly the impact factor k_p, the determination of which is delicate. In order to make allowance for the natural discontinuous contact when running, only the positive part of the transform is retained, which may be written, with the previous notations:

$$F(t)=G_0+\sum_{i=1}^{n} G_i \sin 2\pi i f_m t \; for \; (j-1)T_m \le t \le (j-\tfrac{1}{2})T_m$$

$$F(t)=0 \; for \; (j-\tfrac{1}{2})T_m < t \le jT_m$$

In this case, the phase shifts are assumed to be negligible and the amplitudes of the first three harmonics are average values, these coefficients being, in strict logic, as k_p, a function of the frequency of running (figure 2.9) :
$G_1 = 1.6\ G_0$; $G_2 = 0.7\ G_0$; $G_3 \approx 0.2\ G_0$.

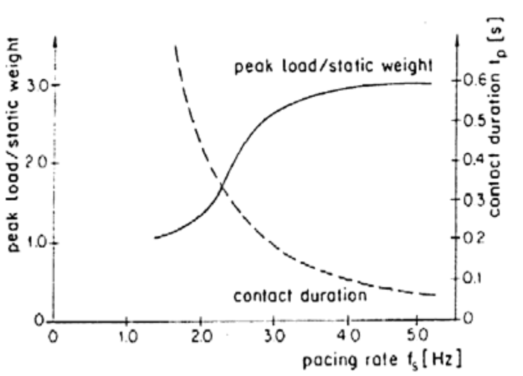


Figure 2.8: Change in the period of contact according to frequency

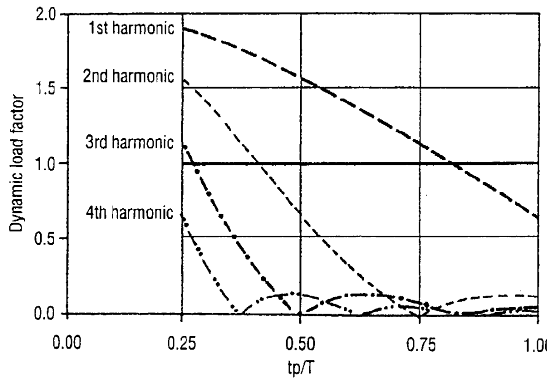


Figure 2.9: Amplitude of the various harmonics

On figure 2.10 are represented (still over 1 second with G_0 = 700 N, but at a frequency f_m = 3 Hz), on the one hand, a semi-sinus approximation and, on the other hand, the Fourier series transform, taking into account 1 to 3 harmonics. We can see that the various graphs are close. More precisely, a semi-sinus approximation leads to a weaker amplitude signal (with the value of k_p used) than taking only one harmonic into account in the Fourier series.

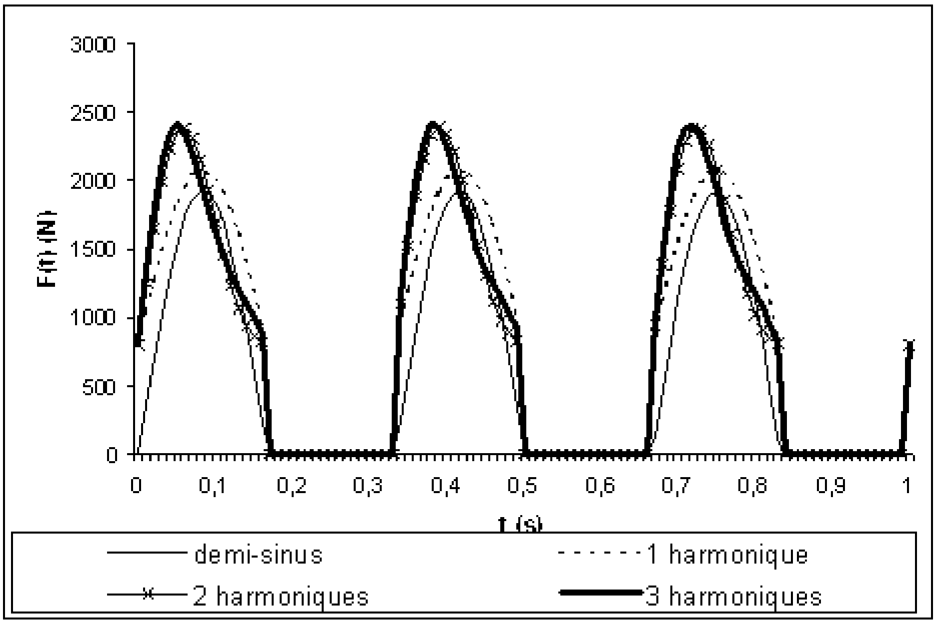


Figure 2.10: running – vertical component (f_m = 3 Hz)

This load according to time can be broken down into a Fourier series, revealing a constant section at about 1250 N and a varying section of the same frequency as the running for the first harmonic and of amplitude 1250 N. It is this value that could be used if a specific analysis of runners had to be made. This guide does not cover specific load cases of runners on a footbridge, as it is considered that the effects of a crowd of pedestrians are clearly less favourable.

2.2.2 - Horizontal component

To our knowledge, no measurements have been taken of the horizontal component during a running race, either of its longitudinal projection, or of its transverse projection. However, it is reasonable to think, on the one hand, that, during a race, the transverse component (that to which the public is most sensitive) has a low relative amplitude compared with the vertical component (the race appearing to be a more "directed" progress), whereas the longitudinal component will be larger (larger motor force). In addition, as for walking, it is logical to estimate that the frequency of the transverse component will be half the vertical component, whereas that of the longitudinal component will be of the same order of magnitude.

3. Appendix 3: Damping systems

3.1 - Visco-elastic dampers

The use of visco-elastic materials for controlling vibrations dates back to the 1950s, when they were used to limit fatigue damage caused by vibrations on aircraft frames. The application to civil engineering constructions dates from the 1960s.

3.1.1 - Principle

The visco-elastic materials used are typically polymers that dissipate energy by working in shear. Figure 3.1 shows a visco-elastic damper formed from layers of visco-elastic materials between metal plates. When this type of device is installed on a construction, the relative displacement of the outer plates in respect of the central plate produces shear stresses in the visco-elastic layer, which dissipates the energy.

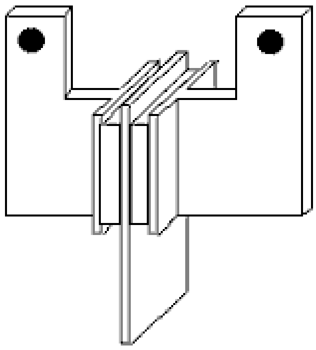


Figure 3.1 – Visco-elastic damper

Under a harmonic load of frequency ω, the strain $\gamma(t)$ and the shear stress $\tau(t)$ are also at the frequency ω, but, in general, out of phase:

$$\begin{cases}\gamma(t) = \gamma_0 \sin(\omega\, t) \\ \tau(t) = \tau_0 \sin(\omega\, t + \varphi)\end{cases} \qquad \textit{(Eq. 3.1.)}$$

The parameters γ_0, τ_0 and φ depend, in general, on the frequency ω. The shear stress may, however, be re-written:

$$\begin{aligned}\tau(t) &= \gamma_0\left(\frac{\tau_0}{\gamma_0}\cos\varphi\,\sin(\omega\, t) + \frac{\tau_0}{\gamma_0}\sin\varphi\,\cos(\omega\, t)\right) \\ &= \gamma_0\big(G_1(\omega)\sin(\omega\, t) + G_2(\omega)\cos(\omega\, t)\big)\end{aligned} \qquad \textit{(Eq. 3.2.)}$$

By replacing the terms $\sin(\omega\, t)$ and $\cos(\omega\, t)$ by $\gamma(t)/\gamma_0$ and $\dot{\gamma}(t)/(\omega\,\gamma_0)$, the following stress-strain ratio is obtained:

$$\tau(t) = G_1(\omega)\,\gamma(t) + \frac{G_2(\omega)}{\omega}\dot{\gamma}(t) \qquad \textit{(Eq. 3.3.)}$$

which defines an ellipse (Figure 3.2), the area of which is the energy dissipated by the material per unit of volume and per cycle of oscillation.

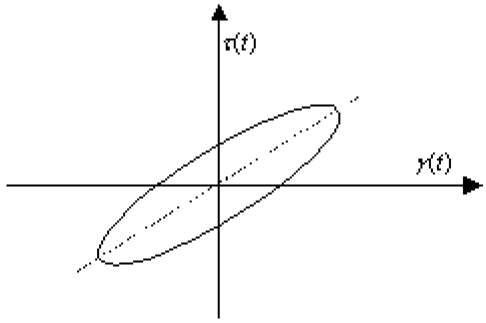


Figure 3.2 – Stress-strain diagram

This energy is given by:

$$E = \int_0^{2\pi/\omega} \tau(t)\,\dot{\gamma}(t)\,dt = \pi\,\gamma_0 \tau_0 \sin\varphi \qquad \textit{(Eq. 3.4.)}$$

The equation *(3.3)* reveals an initial term in phase with the displacement, representing the elastic modulus; the second term describes the dissipation of energy. The damping $G_2(\omega)/\omega$ leads to expressing the damping factor by:

$$\xi = \frac{G_2(\omega)}{\omega}\,\frac{\omega}{2G_1(\omega)} = \frac{\tan\varphi}{2} \qquad \textit{(Eq. 3.5.)}$$

The parameters $G_2(\omega)$ and $\tan\varphi$ determine completely the behaviour of the visco-elastic damper under harmonic excitation. The term $\tan\varphi$ is called the *dissipation factor*. These factors vary not only according to the frequency, but are also function of the ambient temperature. As the dissipation of energy is in the form of heat, the level of performance of the visco-elastic dampers must also be evaluated in relation to the variation in internal temperature that appears when they are working.

3.1.2 - Layout and design of dampers

The layout of the dampers is an essential parameter in their effectiveness in relation to the dissipation of energy. However, technical constraints may prevent the most appropriate places from being selected. The effectiveness of a visco-elastic damper is measured by its capacity to work in shear; it is, therefore, recommended that the dampers be set out in such a way that this condition can be checked.
This type of damper has been little used for footbridges.

3.2 - Viscous dampers

3.2.1 - Principle

Dry or visco-elastic friction dampers use the action of solids to dissipate the vibratory energy of a construction (Figure 3.3). It is, however, also possible to use a fluid.
The first device to come to mind is one derived from a dashpot. In such a device, the dissipation takes place by the conversion of the mechanical energy into heat, using a piston that deforms a very viscous substance, such as silicone, and displaces it. Another family of dampers is based on the flow of a fluid in a closed container. The piston is not limited to deforming the viscous substance, but forces the fluid to pass through calibrated orifices, fitted or not with simple regulation devices. As in the previous case, the dissipation of the energy

leads to heat being given off. Very high levels of energy dissipation can be reached, but require adequate technological devices.
The main difference between these two technologies is as follows. In the case of a pot or wall damper, the dissipative force will depend on the viscosity of the fluid, whereas, in the case of an orifice damper, this force is due mainly to the density of the fluid. Orifice dampers will therefore be more stable in respect of variations in temperature than pot dampers or wall dampers.
In the case of pedestrian footbridges, the movements are very weak and dampers must be selected that will be effective even for small displacements, in the order of one millimetre. Because of the compressibility of the fluid, friction in the joints, and tolerance in the fixings, this is not easy to achieve.
Viscous dampers must be located between two points on the construction having a differential displacement in respect of each other. The larger these differential displacements, the more effective the damper will be. In practice, these dampers will be located either on an element connecting a pier and the deck a few metres away from the pier (for horizontal or vertical vibrations), or on the horizontal bracing elements of the deck (for horizontal vibrations).

Figure 3.3 – Example of viscous damper

3.2.2 - Laws of behaviour of viscous dampers

As part of the study of the behaviour of a construction, it is necessary to have a macroscopic behaviour model of the damper. For this, it is traditional to use a force-displacement law as described by a differential equation of order κ (Ref. [55]) :

$$f(t) + \lambda^{\kappa} \frac{d^{\kappa} f}{dt^{\kappa}}(t) = C_0 \frac{dx}{dt}(t) \qquad \textit{(Eq. 3.6.)}$$

in which $f(t)$ is the force applied to the piston, and $x(t)$ the resultant displacement of the piston. The parameters C_0, λ, κ represent respectively the damping factor at zero frequency, the relaxation time and the order of the damper. These parameters are generally determined experimentally, although approximations do exist to estimate them analytically on the basis of the characteristics of the materials (Ref. [56]).
Notes:
When $\lambda \equiv 0$, the case for a linear viscous damper is made. It should be pointed out that it is often the most-used, as, obviously, it simplifies an analysis of the behaviour of the construction.
Another formulation may be used:

$$f(t) = C \left(\frac{dx}{dt}(t) \right)^{\alpha} \qquad \textit{(Eq. 3.7.)}$$

α is a coefficient generally between 0.1 and 0.4.

Note that, for certain fluids, the kinematic viscosity also varies according to the rate of shear of the fluid, which is itself proportional to the speed gradient.
Viscous dampers were installed on the Millennium footbridge in London.

3.3 - Tuned mass dampers (TMD)

A tuned mass damper (abbreviated to TMD) comprises a mass connected to the construction by a spring and by a damper positioned in parallel. This device enables the vibrations in a construction in a given vibration mode, under the action of a periodic excitation of a frequency close to the natural frequency of this mode of vibration of the construction, to be reduced substantially.
The first developments of TMDs applied essentially to mechanical systems for which a frequency of excitation is in resonance with the basic frequency of the machine.

3.3.1 - Principle

Consider an oscillator with 1 degree of freedom, subjected to a harmonic force $f(t)$. The response of this oscillator may be reduced in amplitude by the addition of a secondary mass (or TMD) that has a relative movement in respect of the primary oscillator. The equations describing the relative displacement of the primary oscillator in respect of the TMD:

$$\begin{aligned} &M\,\ddot{y}_1(t) + C\,\dot{y}_1(t) + K\,y_1(t) = c\,\dot{y}_2(t) + k\,y_2(t) + f(t) \\ &m\,\ddot{y}_2(t) + c\,\dot{y}_2(t) + k\,y_2(t) = -m\,\ddot{y}_1(t) \end{aligned} \qquad \textit{(Eq. 3.8.)}$$

which is reduced to the single equation:

$$(M+m)\,\ddot{y}_1(t) + C\,\dot{y}_1(t) + K\,y_1(t) = -m\,\ddot{y}_2(t) + f(t) \qquad \textit{(Eq. 3.9.)}$$

3.3.2 - Den Hartog's solution

In the case where the primary oscillator has zero damping ($C = 0$), it is possible to determine the optimum characteristics of the TMD for a harmonic excitation $f(t)$ of amplitude f_0 and of pulsation ω. This optimum solution is the solution deduced by Den Hartog in his first studies. For a harmonic excitation, the impact factor A, defined as the ratio between the maximum dynamic amplitude $y_{1\max}$ and the static displacement $y_{1\text{stat}}$, is expressed by:

$$A = \frac{y_{1\max}}{y_{1\text{stat}}} = \sqrt{\frac{\left(\alpha^2-\beta^2\right)^2 + \left(2\xi_{\text{ada}}\alpha\beta\right)^2}{\left[\left(\alpha^2-\beta^2\right)\left(1-\beta^2\right)-\alpha^2\beta^2\mu\right]^2 + \left(2\xi_{\text{ada}}\alpha\beta\right)^2\left(1-\beta^2-\beta^2\mu\right)^2}} \qquad \textit{(Eq. 3.10.)}$$

where:

$$\beta = \frac{\omega}{\omega_{\text{osc}}}\ ;\ \alpha = \frac{\omega_{\text{ada}}}{\omega_{\text{osc}}}\ ;\ \omega_{\text{ada}}^2 = \frac{k}{m} \qquad \textit{(Eq. 3.11.)}$$

$$\omega_{\text{osc}}^2 = \frac{K}{M}\ ;\ \xi_{\text{ada}} = \frac{c}{2m\omega_{\text{ada}}}\ ;\ \mu = \frac{m}{M} \qquad \textit{(Eq. 3.12)}$$

The dynamic amplification factor A therefore depends on four parameters: $\mu, \alpha, \beta, \xi_{ada}$. Figure 3.4 gives the changes of A according to the frequency ratio β, for various damping factors and for $\alpha = 1$ (agreement of frequencies) and $\mu = 0{,}05$. This figure shows that, for the case where the damping ξ_{ada} is zero, the response amplitude has two resonance peaks. Contrary to when the damping tends towards infinity, the two masses are virtually fused, so as to form a single oscillator of mass $1{,}05\,M$ with an infinite amplitude at the resonance frequency. Between these extreme cases, there is a damping value for which the resonance peak is minimal.

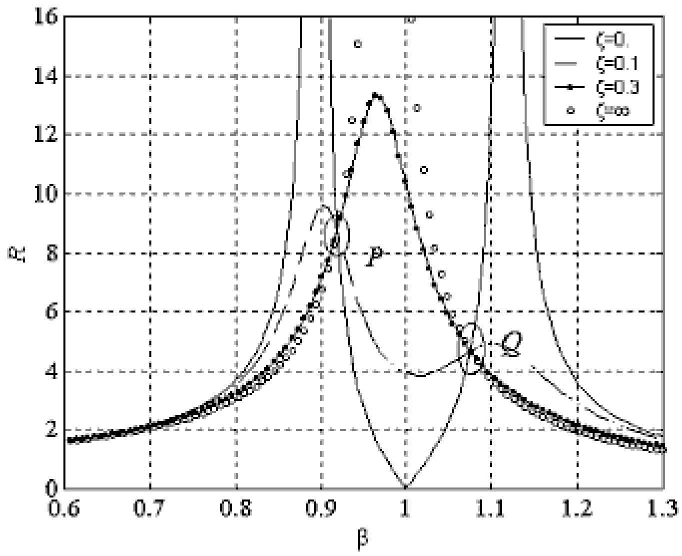


Figure 3.4 – Impact factor depending on β
($\mu = 0{,}05$ et $\alpha = 1{,}0$)

The objective of a tuned mass damper consists, therefore, of bringing the resonance peak down as low as possible. Figure 3.4 shows the independence in relation to ξ_{ada} of two points (*P* and *Q*) on the graphs $A = f(\beta)$. The minimum amplitude of the resonance is thus obtained by selecting the ratio α such that these two points are of equal amplitudes. This optimum ratio is thus given by the expression:

$$\alpha_{opt} = \frac{1}{1 + m/M} \qquad \text{(Eq. 3.13)}$$

and the amplitudes at points *P* and *Q* are:

$$R = \sqrt{1 + \frac{2}{\mu}} \qquad \text{(Eq. 3.14)}$$

The optimum frequency f_{opt} and the optimum damping factor ξ_{opt} can be determined for the tuned mass damper [(1)], in the case where the construction does not have its own damping: this is *Den-Hartog's solution* (Ref. [51]):

$$f_{opt} = \frac{1}{1 + \mu} f_s \qquad \text{(Eq. 3.15)}$$

$$\xi_{opt} = \sqrt{\frac{3}{8}\frac{\mu}{(1 + \mu)^3}} \qquad \text{(Eq. 3.16)}$$

3.3.3 - General solution

A more detailed analysis has been carried out by Warburton (Ref. [54]) in order to determine the optimum characteristics of the TMD for several types of excitations (in the presence of weak structural damping). The analysis proposed by Warburton also enables the results to be

[(1)] The optimum frequency is very slightly less than that of the construction, the added mass always being in the order of a few hundredths of the mass of the construction.

used for primary oscillators with one degree of freedom, to cases with several degrees of freedom. These analogies are based on a breakdown on the basis of the standardised natural modes of the construction. Optimality is not therefore sought on a given mode (modal component), but on a generalised coordinate (displacement or derivatives). The reader is therefore referred to reference 54 for further details on these analogies. In addition, this section will be limited to dealing only with the case of oscillators with 1 degree of freedom.
The optimum characteristics of Den Hartog's tuned mass damper for a harmonic excitation have been established, minimising the impact factor A. This is the very principle of the determination of an optimum TMD: first defining a criterion of optimality A, then seeking the solutions minimising this criterion. Apart from the impact factor, numerous other criteria can be selected, such as the minimisation of the displacement of the construction, of the displacement of the damper, of the acceleration of the construction or of forces in the construction, etc., all for various types of excitations. In particular, when the excitation is random (the primary system is subjected to a random force – in other words, in our case, the crowd), the optimum parameters α_{opt} and ξ_{opt} are given by the expressions:

$$\alpha_{opt} = \frac{\sqrt{1+\frac{\mu}{2}}}{1+\mu} \qquad \text{(Eq. 3.17)}$$

$$\xi_{opt} = \sqrt{\frac{\mu\left(1+\frac{3\mu}{4}\right)}{4(1+\mu)\left(1+\frac{\mu}{2}\right)}} \qquad \text{(Eq3.18)}$$

3.3.4 - Case of a damped primary oscillator

In the case where the construction has natural damping, the theoretical optimum frequency is very slightly weaker than that given by the above formula (valid for a harmonic excitation), which is nevertheless sufficient in usual cases (Ref. [53]) :

$$\begin{aligned} \tilde{\alpha}_{opt} &= \alpha_{opt} - \left(0{,}241 + 1{,}7\mu - 2{,}6\mu^2\right)\xi_{osc} - \left(1{,}0 - 1{,}9\mu + \mu^2\right)\xi_{osc}^2 \\ \tilde{\xi}_{opt} &= \xi_{opt} + \left(0{,}13 + 0{,}12\mu + 0{,}4\mu^2\right)\xi_{osc} - \left(0{,}01 + 0{,}9\mu + 3\mu^2\right)\xi_{osc}^2 \end{aligned} \qquad \text{(Eq. 3.19)}$$

in which ξ_{osc} is the damping factor of the primary mass. These expressions show less than 1% error for $0.03<\mu<0.40$ and $0.0<\xi_{osc}<0.15$.

The effectiveness of a tuned mass damper is much more sensitive to the natural frequency of the added mass, than to the value of the added damping. This is why it must be possible to adjust the natural frequency of the mass when fixing the device, so as fine-tune it to the actual natural frequency of the construction. This is generally done by varying the value of the mass itself, which is much easier to adjust than the stiffness of the spring.

3.3.5 - Examples

There are various types of tuned mass dampers. The most typical type consists of a mass connected to the construction by means of vertical coil springs and one or more hydraulic or pneumatic dampers. These TMDs may be linked together to damp torsion vibrations. Figure 3.5 gives an overview of such a device.

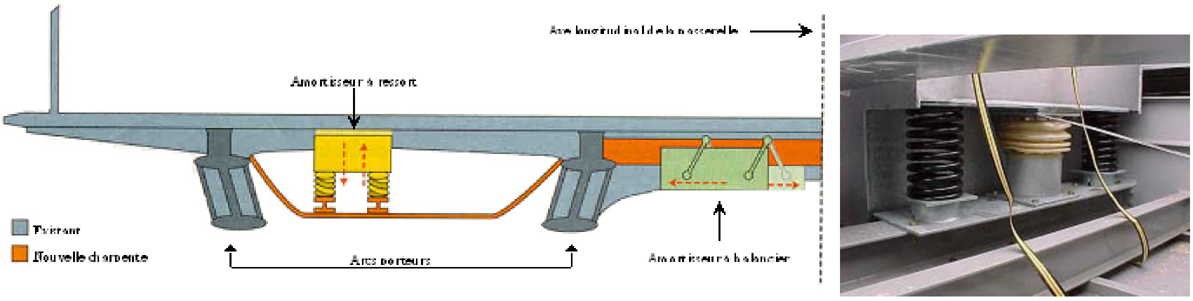
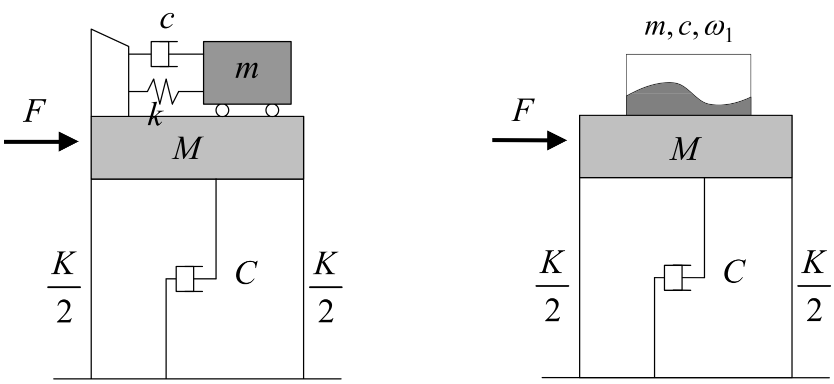


Figure 3.5 – Tuned mass damper for the Solferino footbridge

When the vibration to be damped is horizontal, a device consisting of a mass attached to the bottom of a pendulum, associated with a horizontal hydraulic damper, may be considered.

3.4 - Tuned liquid dampers

In the technology of tuned mass dampers, a second mass is attached to the construction using struts and dampers (figure 3.6a). Another category of dampers consists of replacing the mass, strut and damper by a container filled with a liquid. As for a traditional TMD, the liquid acts as the secondary mass and the damping is provided by friction with the walls of the container. As the action of gravity comprises a recall mechanism, the secondary system (figure.3.6b) thus formed has characteristic frequencies that can be tuned to optimise one performance criterion.
The first TLD prototype was proposed in the 1900s by Frahm to control rolling in ships. Since the 1970s, these dampers have been installed on satellites to reduce long period vibrations. But it has only been since the 1980s that building applications have been considered.



(a) TMD with mass (b) TLD

Figure 3.6 – Comparison between TMD with mass and TLD with fluid

The principles used for the sizing of TMDs also apply to TLDs. However, although the parameters of a TMD can be optimised and analytical formulae provided, the non-linear response of the fluid in movement in a container makes such optimisation very difficult. The response of the "container/construction" system is thus dependant on the amplitude of the movements.

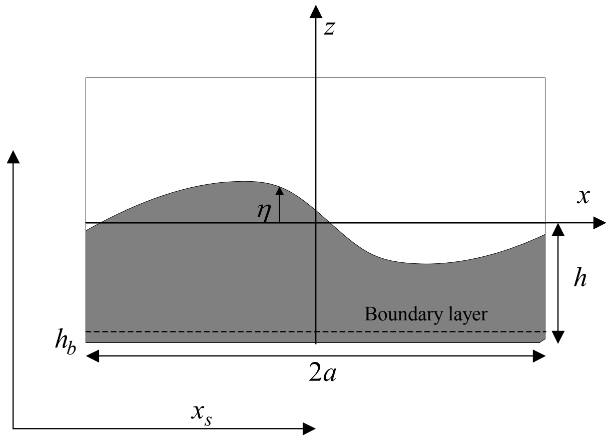


Figure 3.7 – Description of a TLD

Let us consider a rectangular container of length *2a* filled with a liquid of viscosity ν and of average depth *h*. The fluid is assumed to be incompressible and irrotational. The container is subjected to a horizontal displacement $x(t)$. It is assumed that the free surface remains continuous (no waves breaking) and that the pressure $p(x,z,t)$ is constant over this free surface (figure 3.7). According to the linear theory of the boundary layer, the natural vibration frequency of the fluid is:

$$\omega_{adaf} = \sqrt{\frac{\pi g}{2a}\tanh\left(\frac{\pi h}{2a}\right)}; \qquad \textit{(Eq. 3.20)}$$

The damping factor may be approached by the expression (Ref. [57]) :

$$\xi_{adaf} = \frac{1}{\sqrt{2}h}\sqrt{\frac{\nu}{\omega_{adaf}}}\left(1+\frac{h}{b}\right); \qquad \textit{(Eq. 3.21)}$$

in which b is the width of the tank. The equations of the coupled system are written in exactly the same way as equations *(3.8.)* and *(3.9.)* with $m = \rho_{adaf} 2abh$, $c = 2m\xi_{adaf}\omega_{adaf}$, $k = m\omega_{adaf}^2$, in other words:

$$\begin{bmatrix} 1+\mu & \mu \\ \mu & \mu \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} 2\xi_{osc}\omega_{osc} & 0 \\ 0 & 2\mu\xi_{adaf}\omega_{adaf} \end{bmatrix}\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{bmatrix} \omega^2{}_{osc} & 0 \\ 0 & \omega^2{}_{adaf} \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} \frac{f(t)}{m} \\ 0 \end{Bmatrix}; \qquad \textit{(Eq. 3.22)}$$

This equation shows the analogy that can exist between a tuned mass damper and a tuned liquid damper. This analogy only makes sense subject to the validity of the expressions (3.20) and (3.21) selected as equivalent pulsation and damping factor. In reality, the resolution of the coupled problem is more complex due to the fluid nature of the secondary oscillator.
As opposed to numerous high-rise towers, to our knowledge, no footbridge has had tuned fluid dampers installed. They were, however, considered for the Ikuchi bridge (Japan), in order to dampen the horizontal vibrations of the pylons. The total fluid mass was 4770 kg and the tuning frequency was 0.255 Hz.

3.5 - Comparative table

Type of damper	Field of use	Advantages	Disadvantages
Visco-elastic	Very little used	Damps several modes	Needs to be fixed to work in shear

Viscous pot or wall dampers	Little used	Damps several modes	Temperature-sensitive, non-linear calculation
Viscous orifice dampers	Little used	Independent of temperature, damps several modes	Non-linear calculation
Tuned mass dampers	Widely used	Easy to size	Additional mass to be considered, damps a given mode, needs adjustment in frequency
Tuned liquid dampers	Very little used		"Innovative", additional mass to be considered, damps a given mode, needs adjustment in frequency

4. Appendix 4: Examples of footbridges

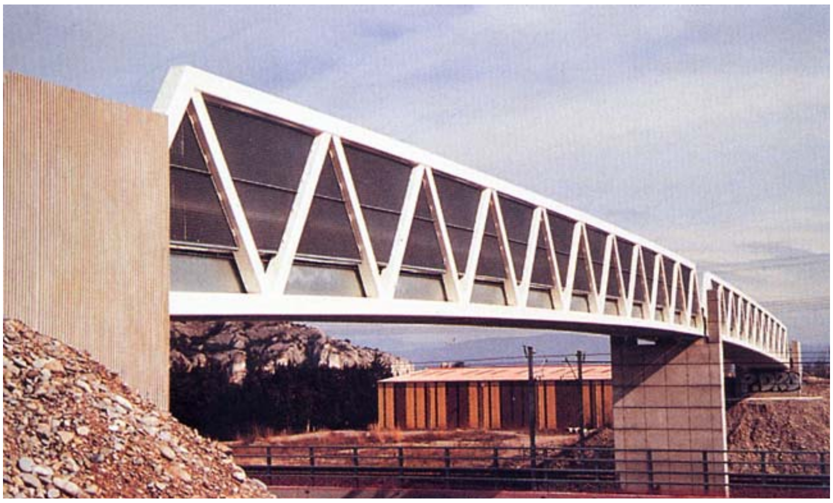
We will be reviewing several different designs of footbridges recently constructed, from the main types of constructions. We will be presenting a construction with lateral beams, a steel box-girder construction with orthotropic deck, a construction with a ribbed slab, a bow-string arch with orthotropic deck, a suspended construction with one steel mast, a steel lattice arch, a cable-stayed construction. We will indicate the main features of each of the bridges and its method of construction, and we will give the results of the dynamic studies and, as applicable, the results of tests.

4.1 - Warren-type lateral beams: Cavaillon footbridge

The bridge is a Warren-type construction with lateral beams 3.20 m high; it comprises two independent bays of 49.7 m span. The deck consists of a reinforced concrete slab bearing on floor beams 2.26 m apart. The functional width is 3.00 m. The works were carried out in 2000 - 2001. The steel construction was erected with a crane, the mass of the framework is 18.8 t per bay.
A modal analysis made using an analytical calculation gave the following results:

Mode 1 Vertical bending 1.95 Hz

This frequency was calculated without taking the mass of the pedestrians into account.



Photograph 4.1 – Cavaillon footbridge

4.2 - Steel box-girder: Stade de France footbridge

The bridge is a box-girder bridge with double intermediate supports, of a total length of 180 m, with a central bay of 64 m span, and end bays of 54 and 50 m. The structure of the deck is formed from a steel box girder with orthotropic topping. The functional width is 11.00 m. The works were carried out in 1997 - 1998. The construction was erected with a crane; the total mass of the framework is 595.0 t.
The modal analysis carried out using finite element design software gave the following results:

Mode 1 Vertical bending central bay 1.97 Hz

Mode 2	Vertical bending of end bay 1	2.06 Hz
Mode 3	Vertical bending of end bay 2	2.20 Hz

The dynamic study gave the following results:

1 pedestrian deflection 2.7 mm acceleration 0.4 m/sec^2

640 pedestrians 25 in phase deflection 70 mm acceleration 10 m/sec^2

The dynamic study with damper of 2.4 t mass:

640 pedestrians 25 in phase deflection 6.5 mm acceleration 1 m/sec^2

Photograph 4.2 – Stade de France footbridge

4.3 - Ribbed slab: Noisy-le-Grand footbridge

The bridge is a ribbed slab in pre-stressed concrete of varying depth of a total length of 88 m, with a central bay of 44 m span, and end bays of 22 m. The depth of the slab varies from 1 m at the summit to 3.05 m at the piers. The functional width is 5 m. The works were carried out in 1993 - 1994. The two beams comprising the structure were constructed perpendicularly to their final position and positioned over the A4 motorway by rotation; the total mass of the deck is 860.0 t.

The modal analysis carried out using finite element design software gave the following results:

Mode 1	Vertical bending	1.65 Hz
Mode 2	Axi-symmetrical bending	3.48 Hz
Mode 3	Lateral bending	4.86 Hz

Photograph 4.3 – Noisy-le-Grand footbridge

4.4 - Bow-string arch: Montigny-lès-Cormeilles footbridge

The bridge is a bow-string with polygonal arch of 55 m span. The structure of the deck comprises two lateral steel tubes connected by an orthotropic topping. The functional width is 3.50 m. The works were carried out in 1998 - 1999. The construction was erected with a crane; the total mass of the framework is 85.0 t.
A modal analysis carried out using finite element calculation software gave the following results, with the tubes of the tie filled with cement, and the mass of the bridge being increased by the mass of the pedestrians corresponding to a(l):

Mode 1	Vertical bending	2.51 Hz
Mode 2	Anti-symmetrical bending	2.52 Hz
Mode 3	Lateral bending of deck	2.62 Hz

Comment: without the pedestrians, the frequencies increase by 0.5 Hz.

Photograph 4.4 – Montigny-lès-Cormeilles footbridge

4.5 - Suspended construction: Footbridge over the Aisne at Soissons

The bridge is a suspension bridge of 60 m span. The structure of the deck is formed from a steel box-girder suspended from a mast consisting of four inclined tubes joining at mid-span. The functional width is 3.00 m. The works were carried out in 2000. The structure was erected with a crane on temporary legs before the mast was erected; the total mass of the steel framework is 105.0 t.
The modal analysis carried out using finite element design software gave the following results:

Mode 1	Lateral bending of deck	1.10 Hz
Mode 2	Lateral bending of the arches	2.80 Hz
Mode 3	Vertical bending of the deck	3.10 Hz

Photograph 4.5 – Soissons footbridge

4.6 - Steel arch: Solferino footbridge

The bridge is a steel arch of 106 m span. The structure of the arch is formed from two double parabolic arches as a ladder beam, linked by cross-pieces supporting a lower deck. The double upper deck is supported on A-frames and braces bearing on the two arches. The functional width varies from 12 m to 14.80. The works were carried out in 1998 - 1999. The structure was erected with a crane on temporary legs; the total mass of the framework is 900.0 t.
The modal analysis carried out using finite element design software gave the following results:

Mode 1	Lateral swing	0.71 Hz
Mode 2	Axi-symmetrical vertical bending	1.03 Hz
Mode 3	Torsion-bending	1.37 Hz
Mode 4	Vertical bending	1.66 Hz

Mode 5	Torsion-swing	1.66 Hz

The dynamic tests and measurements gave the following frequencies empty (without damper):

Mode 1	Lateral swing	0.81 Hz
Mode 2	Axi-symmetrical vertical bending	1.22 Hz
Mode 3	Torsion-bending	1.59 Hz
Mode 4	Vertical bending	1.69 Hz
Mode 5	Central torsion-swing	1.94 Hz
Mode 6	Central torsion-swing	2.22 Hz
Mode 7	Bending-torsion	3.09 Hz

The damping for the empty structure varied from 0.3% to 0.5%.
The dynamic tests and measurements gave the following accelerations:

16 pedestrians balance mode 1	acceleration 0.5 m/sec^2
16 pedestrians walking mode 6	acceleration 2.0 m/sec^2
16 pedestrians running mode 7	acceleration 2.5 m/sec^2

The damping with 16 pedestrians on the footbridge varied from 0.4% to 0.8%.

106 pedestrians balance mode 1	acceleration 1.5 m/sec^2
106 pedestrians running mode 7	acceleration 5.7 m/sec^2

The damping with more than 100 pedestrians on the footbridge varied from 0.7% to 1.6%.

Dampers

6 pendular systems supporting masses of 2.5 t and 1.9 t allow damping of 3.9% in respect of mode 1 (lateral swing) to be achieved.

8 mass/spring systems supporting masses of 2.5 t allow damping of 2.75% in respect of modes 5 and 6 (central torsion-swing) to be achieved.

Photograph 4.6 – Solferino footbridge

4.7 - Cable-stayed construction: Pas-du-lac footbridge at St Quentin

The bridge is a dissymmetric cable-stayed bridge of a total length of 188 m, with, on one side, two bays with spans of 68 m and 36 m, and, on the other, two bays, each of 42 m span. The structure of the deck is formed from two steel beams, suspended from a single pylon, linked by floor beams. The functional width is 2.50 m. The works were carried out in 1991 - 1992. The construction was positioned with a crane.
The modal analysis carried out using finite element design software gave the following results:

Mode 1	Lateral bending of deck	1.38 Hz
Mode 2	Displacement-bending of deck	1.85 Hz
Mode 3	Bending of pylon	1.92 Hz
Mode 4	Vertical bending of the deck	1.95 Hz

Photograph 4.7 – Pas-du-Lac footbridge

4.8 - Mixed construction beam: Mont-Saint-Martin footbridge

The bridge has a mixed steel-concrete deck with a span of 23 m. The structure of the deck is formed from two slightly cambered steel beams linked by floor beams. The functional width is 2.50 m. The works were carried out in 1996. The structure was erected with a crane; the mass of the framework is 22.0 t.
A modal analysis made using an analytical calculation gave the following results:

Mode 1	Vertical bending	2.15 Hz
Mode 2	Vertical bending	3.99 Hz
Mode 3	Lateral swing	4.50 Hz

5. Appendix 5: Examples of calculations of footbridges.

This section sets out a full study of two standard footbridges based on actual examples, together with a sensitivity study of the natural frequencies of typical footbridges.

The two complete examples of dynamic calculation have been carried out using the methodology of the guide, taking into account the various categories of traffic. If the results led to unacceptable accelerations, the characteristics of these constructions were modified to improve their dynamic behaviour and to try and satisfy comfort conditions.

5.1 - Examples of complete calculations of footbridges

5.1.1 - Warren-type lateral beam footbridge

The first footbridge studied is a construction with a mixed steel-concrete framework forming one independent bay with a span of 38.85 m. The longitudinal profile is curved to a radius of 450 metres.

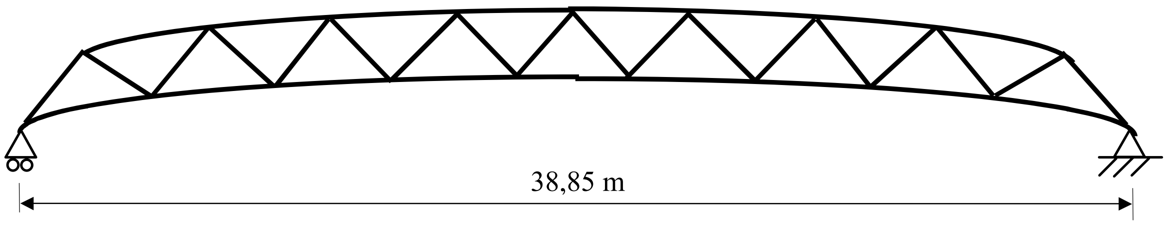


Figure 5.1: Warren-type lateral beam footbridge

The framework is formed from two triangulated lateral beams. These beams, of a constant depth of 1.215 m, are linked by floor beams located at the level of the bottom member. A pre-cast reinforced concrete slab, 10 cm thick, bears on these floor beams.

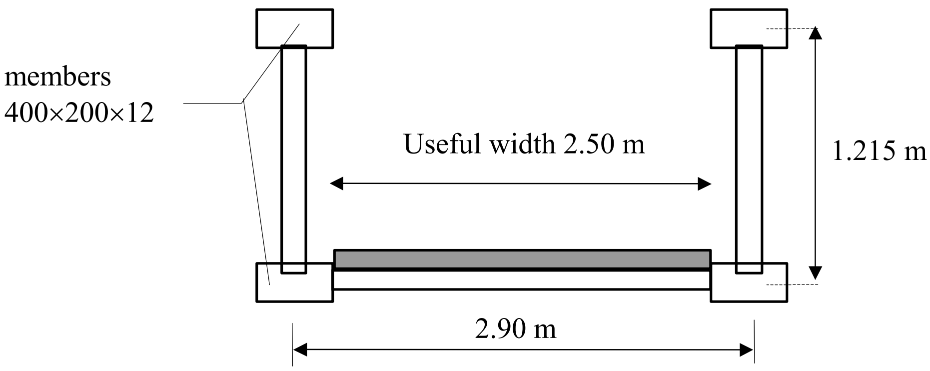


Figure 5.2: Cross-section of the lateral beam footbridge

The distance between the centre lines of the beams is 2.90 m, giving a width of passage for pedestrians of 2.50 m.

<u>Characteristics of the deck</u>

The moment of inertia is calculated taking into account the reinforced concrete slab with a homogenisation coefficient of 6, and the mass of the deck is calculated taking into account the floor beams and the reinforced concrete slab.

Moment of inertia of the deck: $I = 0{,}030\,m^4$
Natural linear density of the deck: $m = 1456\,kg/m$
Young's modulus of the steel: $E = 210\times10^9\ N/m^2$

5.1.1.1 Class III

First of all, we will consider class III, in other words a normally-used footbridge that can sometimes be crossed by large groups, but never loaded over its whole area.

5.1.1.1.1 Calculation of natural modes

The natural frequencies are equal to: $f_n = \dfrac{n^2\pi}{2L^2}\sqrt{\dfrac{EI}{\rho S}}$

ρS is the linear density of the deck increased by the linear density of the pedestrians, which is calculated for each crowd density, depending on the class of the footbridge.

For class III, we are interested in a sparse crowd, where the density *d* of the crowd is equal to 0.5 pedestrians/m^2.
The number of pedestrians on the footbridge is: n_p=0.5×38.85×2.5=48.6
The total mass of the pedestrians is: 70×48.6=3402 kg
The linear density of the pedestrians is: m_p=3402/38.85=87.6 kg/m
The linear density is: ρS=1456+87.6=1534.6 kg/m
For the first mode, the high-bond and low bond frequencies are equal to:

$$f_1 = \frac{(1)^2\pi}{2(38.85)^2}\sqrt{\frac{210\times10^9\times0.030}{1456}} = 2.14Hz$$

$$f_1 = \frac{(1)^2\pi}{2(38.85)^2}\sqrt{\frac{210\times10^9\times0.030}{1630.8}} = 2.02Hz$$

For the second mode, the high-bond and low bond frequencies are equal to:

$$f_2 = \frac{(2)^2\pi}{2(38.85)^2}\sqrt{\frac{210\times10^9\times0.030}{1456}} = 8.54\ Hz$$

$$f_2 = \frac{(2)^2\pi}{2(38.85)^2}\sqrt{\frac{210\times10^9\times0.030}{1630.8}} = 8.08\ Hz$$

Only the first mode is likely to cause uncomfortable vibrations.

5.1.1.1.2 Calculation of the dynamic load of the pedestrians

We will calculate the load for the first mode only, with a critical damping ratio of 0.6% (mixed deck).

The surface load to be taken into account for the vertical modes is:

$F_s = d \times (280N) \times \cos 2\pi f_v t \times 10.8 \times \sqrt{\frac{\xi}{n}} \times \psi$

ψ is equal to 1, as the frequency of the first mode, which is 2.08 Hz, is within range 1 (1.7 to 2.1 Hz) with a maximum risk of causing resonance.

We have $10.8\sqrt{\frac{\xi}{n}} = 10.8\sqrt{\frac{0.6/100}{48.6}} = 0.120$

The surface load is equal to:

$F_s = 0.5 \times 280 \times \cos(2\pi \times 2.08 \times t) \times 0.120 \times 1 = 16.8 \times \cos(2\pi \times 2.08 \times t)\, N/m^2$

The linear load is equal to:

$F = F_s \times l_p = 16.8 \times 2.5 \times \cos(2\pi \times 2{,}08 \times t) = 42.0 \times \cos(2\pi \times 2.08 \times t)\ N/m$

This load is applied to the whole of the footbridge.

5.1.1.1.3 Calculation of dynamic responses

The calculation of the acceleration to which the construction is subjected gives:

$A_{cc\max} = \frac{1}{2 \times 0.6/100} \frac{4 \times 42.0}{\pi \times 1534.6} = 2.91\, m/s^2$

The maximum acceleration calculated is located within range 4 of the accelerations, in other words at an unacceptable comfort level (acceleration > 2.5 m/sec^2).

5.1.1.2 Class II

We will next consider class II, in other words an urban footbridge linking populated zones, subjected to a high level of traffic and likely, on occasions, to be loaded over its whole area.

5.1.1.2.1 Calculation of natural modes

For class II, we are interested in a dense crowd, where the density *d* of the crowd is equal to 0.8 pedestrians/m^2.

The number of pedestrians on the footbridge is: $n_p = 0.8 \times 38.85 \times 2.5 = 77.7 = 78$

The total mass of the pedestrians is: $70 \times 78 = 5460\, kg$

The linear density of the pedestrians is: $m_p = 5460/38.85 = 140.5\, kg/m$

The linear density is: $\rho S = 1456 + 140.5 = 1596.5\, kg/m$

For the first mode, the high-bond and low bond frequencies are equal to:

$$f_1 = \frac{(1)^2 \pi}{2(38.85)^2}\sqrt{\frac{210 \times 10^9 \times 0.030}{1456}} = 2.14 Hz$$

$$f_1 = \frac{(1)^2 \pi}{2(38.85)^2}\sqrt{\frac{210 \times 10^9 \times 0.030}{1630.8}} = 2.02 Hz$$

For the second mode, the high-bond and low bond frequencies are equal to:

$$f_2 = \frac{(2)^2 \pi}{2(38.85)^2}\sqrt{\frac{210 \times 10^9 \times 0.030}{1456}} = 8.54\ Hz$$

$$f_2 = \frac{(2)^2 \pi}{2(38.85)^2}\sqrt{\frac{210 \times 10^9 \times 0.030}{1630.8}} = 8.08\ Hz$$

Only the first mode is likely to cause uncomfortable vibrations.

5.1.1.2.2 Calculation of the dynamic load of the pedestrians

With 0.6% of critical damping, as in the previous case, we have:

$$10.8\sqrt{\frac{\xi}{n}}=10.8\sqrt{\frac{0.6/100}{48.6}}=0.120$$

The surface load is equal to:

$$F_s=0.8\times280\times\cos(2\pi\times2.04\times t)\times0.095\times1=21.28\times\cos(2\pi\times2.04\times t)\ N/m^2$$

The linear load is equal to:

$$F=F_s\times l_p=21.28\times2.5\times\cos(2\pi\times2.04\times t)=53.2\times\cos(2\pi\times2.04\times t)\ N/m$$

5.1.1.2.3 Calculation of dynamic responses

$$A_{cc\max}=\frac{1}{2\times0.6/100}\frac{4\times53.2}{\pi\times1596.5}=3.53\ m/s^2$$

The maximum acceleration calculated is located within range 4 of the accelerations, in other words at an unacceptable comfort level (acceleration > 2.5 m/sec^2).

5.1.1.3 Class I

We will finally consider class I, in other words an urban footbridge linking zones with high concentrations of pedestrians (presence of a station, for example), or frequently used by dense crowds (demonstrations, tourists, etc.), subjected to a very high level of traffic.

5.1.1.3.1 Calculation of natural modes

For class I, we are interested in a very dense crowd, where the density *d* of the crowd is equal to 1.0 pedestrian/m^2.

The number of pedestrians on the footbridge is: $n_p=1\times38.85\times2.5=97$

The total mass of the pedestrians is: $70\times97=6790\ kg$

The linear density of the pedestrians is: $m_p=6790/38.85=174.8\ kg/m$

The linear density is: $\rho S=1456+174.8=1630.8\ kg/m$

For the first mode, the high-bond and low bond frequencies are equal to:

$$f_1=\frac{(1)^2\pi}{2(38.85)^2}\sqrt{\frac{210\times10^9\times0.030}{1456}}=2.14Hz$$

$$f_1=\frac{(1)^2\pi}{2(38.85)^2}\sqrt{\frac{210\times10^9\times0.030}{1630.8}}=2.02Hz$$

For the second mode, the high-bond and low bond frequencies are equal to:

$$f_2=\frac{(2)^2\pi}{2(38.85)^2}\sqrt{\frac{210\times10^9\times0.030}{1456}}=8.54\ Hz$$

$$f_2=\frac{(2)^2\pi}{2(38.85)^2}\sqrt{\frac{210\times10^9\times0.030}{1630.8}}=8.08\ Hz$$

Only the first mode is likely to cause uncomfortable vibrations.

5.1.1.3.2 Calculation of the dynamic load of the pedestrians

We will calculate the load for the first mode only, with 0.6% of critical damping (mixed deck).

The surface load to be taken into account for the vertical modes is:

$$F_s = d \times (280N) \times \cos 2\pi f_v t \times 1.85 \times \sqrt{\frac{1}{n}} \times \psi$$

ψ is equal to 1, as the frequency of the first mode, which is 2.02 Hz, is within range 1 (1.7 to 2.1 Hz) with a maximum risk of causing resonance.
The surface load equals:

$$F_s = 1.0 \times 280 \times \cos(2\pi \times 2.02 \times t) \times 1.85 \sqrt{\frac{1}{97}} \times 1 = 52.60 \times \cos(2\pi \times 2.02 \times t)\, N/m^2$$

The linear load is equal to:

$$F = F_s \times l_p = 52.60 \times 2.5 \times \cos(2\pi \times 2.02 \times t) = 131.5 \times \cos(2\pi \times 2.02 \times t)\ N/m$$

5.1.1.3.3 Calculation of dynamic responses

We have: $A_{cc\max} = \frac{1}{2 \times 0.6/100} \frac{4 \times 131.5}{\pi \times 1630.8} = 8.55\, m/s^2$

The maximum acceleration calculated is located within range 4 of the accelerations, in other words at an unacceptable comfort level (acceleration > 2.5 m/sec^2).

5.1.1.4 Summary

It can be seen that the accelerations are always higher than 2.5 m/sec², whatever class is selected. It should be pointed out that this example is a particularly unfavourable case, as the first natural frequency is in the middle of the range of maximum risk.

In order to reduce the accelerations obtained, the stiffness of the construction must be increased. To do this, the depth of the triangulated lateral beams can be increased (for example by 20 cm) and the thickness of the sheet metal of the members also increased (for example 14 mm).
We choose both to increase the thickness of the metal in the members and to increase the depth of the beams, and then we recalculate the frequencies, the dynamic loads and the corresponding responses.

5.1.1.5 Stiffening of the structure

The sheet metal of the members is increased from 12 to 14 mm. The depth between centre lines of the members is increased from 1.215 m to 1.415 m.

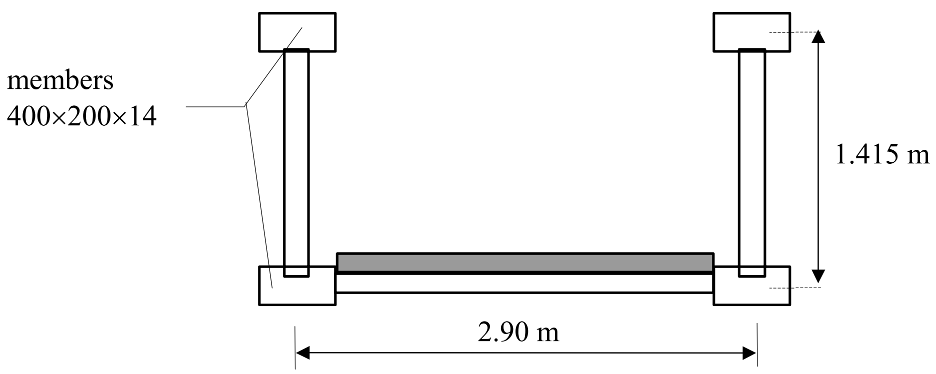


Figure 5.3: Cross-section of the footbridge with members in 14 mm sheet metal and depth of 1.415 m

<u>Characteristics of the deck</u>
The moment of inertia of the deck is modified and equals: $I = 0.045\, m^4$

The natural mass of the deck is slightly increased, but that is not significant.

The frequencies of the first modes are modified in the following way:
Class III: 2.57 Hz
Class II: 2.52 Hz
Class I: 2.50 Hz

The frequency of 2.57 Hz for class III does not lead to any calculation, as this frequency is outside the range 1.7 Hz – 2.1 Hz.

For class I and II, calculations are necessary, but with a coefficient ψ=0.21 for class I and ψ=0.16 for class II.
This leads to the following accelerations:
Acc = 0.55 m/sec² for class II, which is compatible with a medium comfort level, and almost maximum (0.50 m/sec²)
Acc = 1.78 m/sec² for class I, which is compatible with a minimum comfort level (1 – 2.5 m/sec²)
To make this footbridge even more comfortable, it would be possible, for example, to increase the thickness of the sheet metal to 16 mm so that the natural frequencies are greater than 2.6 Hz. The coefficient ψ is then zero. In this case, the second harmonic of the pedestrians must be taken into account, but if it stays around 2.6 Hz, it should not cause any problems.

5.1.2 - Box-girder footbridge

The second footbridge studied is a steel box girder with two bays, each of 40 m, with concrete deck topping.

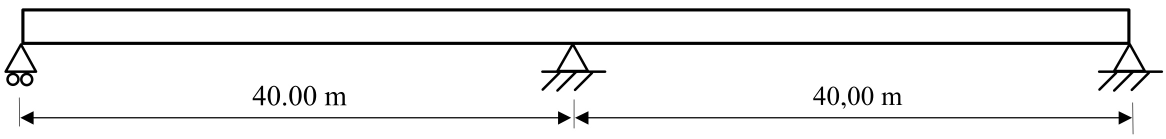


Figure 5.4: Steel box girder footbridge

The framework is formed from a steel box girder of a constant depth of 1 metre. A 10 cm thick pre-cast reinforced concrete slab bears on this girder.

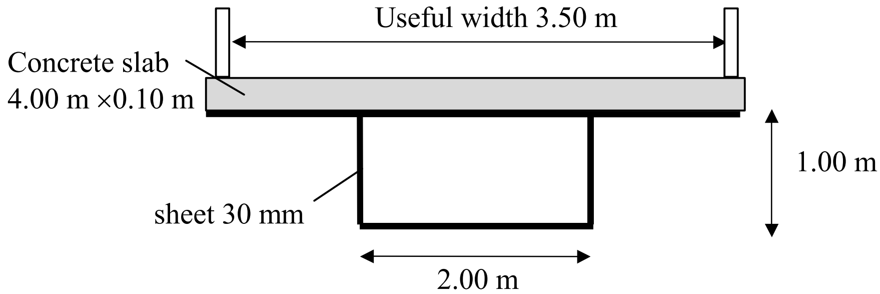


Figure 5.5: Cross-section of a mixed box girder footbridge

The width of the slab is 4.00 m, the width for pedestrian passage is 3.50 m.

Characteristics of the deck

The mass of the deck is calculated taking into account the reinforced concrete slab and the mass of the balustrades. The moment of inertia is calculated taking into account the concrete slab with a homogenisation coefficient of 6.

Moment of inertia of the deck: $I = 0{,}057\ m^4$
Natural linear density of the deck: $m = 3055\ kg/m$
Young's modulus of the steel: $E = 210 \times 10^9\ N/m^2$

5.1.2.1 Class III

First of all, we will consider class III, in other words a normally-used footbridge that can sometimes be crossed by large groups, but never loaded over its whole area.

5.1.2.1.1 Calculation of natural modes

The natural modes of the deck were calculated using the Systus program.

ρS is the total linear density, in other words the linear density of the deck (including the slab), increased by the linear density of the pedestrians, which is calculated for each crowd density, depending on the class of the footbridge.

For class III, we are interested in a sparse crowd, where the density *d* of the crowd is equal to 0.5 pedestrians/m^2.
The number of pedestrians on the two bays is: $n_p = 0.5 \times (2 \times 40) \times 3.5 = 140$
The total mass of the pedestrians is: $70 \times 140 = 9800\ kg$
The linear density of the pedestrians is: $m_p = 9800/80 = 122.5\ kg/m$
The total linear density is: $\rho S = 3055 + 122.5 = 3177.5\ kg/m$
For the first mode, the high-bond and low bond frequencies are equal to:
$f_1 = 1.94\ Hz \quad f_1 = 1.86\ Hz$
For the second mode, the high-bond and low bond frequencies are equal to:
$f_2 = 3.04\ Hz \quad f_2 = 2.92\ Hz$

5.1.2.1.2 Calculation of the dynamic load of the pedestrians

First of all, we calculate the load for the first mode, and a critical damping ratio of 0.6% (mixed deck).
The surface load to be taken into account for the vertical modes is:

$$F_s = d \times (280N) \times \cos 2\pi f_v t \times 10.8 \times \sqrt{\frac{\xi}{n}} \times \psi$$

ψ is equal to 1, as the frequency of the first mode, which is 1.90 Hz, is within range 1 of the frequencies (1.7 to 2.1 Hz).

We have: $10.8\sqrt{\frac{\xi}{n}} = 10.8\sqrt{\frac{0.6/100}{140}} = 0.071$
The surface load is equal to:

$$F_s = 0.5 \times 280 \times \cos(2\pi \times 1.90 \times t) \times 0.071 \times 1 = 9.94 \times \cos(2\pi \times 1.90 \times t)\ N/m_2$$

The linear load is equal to:

$$F = F_s \times l_p = 9.94 \times 3.5 \times \cos(2\pi \times 1.90 \times t) = 34.79 \times \cos(2\pi \times 1.90 \times t)\ N/m$$

5.1.2.1.3 Calculation of dynamic responses

The acceleration under the vertical load is equal to:

$$A_{cc\max}=\frac{1}{2\xi}\frac{4\times F}{\rho S\pi}=\frac{1}{2\times 0.6/100}\frac{4\times 34.79}{3177.5\times\pi}=1.16\ m/s^2$$

The maximum acceleration calculated is located within range 3 of the accelerations, in other words at the minimum comfort level (acceleration between 1 and 2.5 m/sec^2). However, the medium comfort level is almost reached.
For the second mode, the frequency of 2.96 Hz does not impose particular checks.

5.1.2.2 Class II

We will then consider class II.

5.1.2.2.1 Calculation of natural modes

For class II, we are interested in a dense crowd, where the density d of the crowd is equal to 0.8 pedestrians/m^2.
The number of pedestrians on the footbridge is: $n_p=0.8\times(2\times 40)\times 3.5=224$
The total mass of the pedestrians is: $70\times 224=15680\ kg$
The linear density of the pedestrians is: $m_p=15680/80=196\ kg/m$
The total linear density is: $\rho S=3055+196=3251\ kg/m$
For the first mode, the high-bond and low bond frequencies are equal to:
$f_1=1.94\ Hz \quad f_1=1.86\ Hz$
For the second mode, the high-bond and low bond frequencies are equal to:
$f_2=3.04\ Hz \quad f_2=2.92\ Hz$

5.1.2.2.2 Calculation of the dynamic load of the pedestrians

First of all, we calculate the load for the first mode, and a critical damping ratio of 0.6% (mixed deck).
The surface load to be taken into account for the vertical modes is:

$$F_s=d\times(280N)\times\cos 2\pi f_v t\times 10,8\times\sqrt{\frac{\xi}{n}}\times\psi$$

ψ is equal to 1, as the frequency of the first mode, which is 1.879 Hz, is within range 1 of the frequencies (1.7 to 2.1 Hz).

We have $10,8\sqrt{\frac{\xi}{n}}=10,8\sqrt{\frac{0.6/100}{224}}=0.056$

The surface load is equal to:
$F_s=0.8\times 280\times\cos(2\pi\times 1.879\times t)\times 0.056\times 1.0=12.54\times\cos 2\pi 1.879\ N/m^2$
The linear load is equal to:
$F=F_s\times l_p=12.54\times 3.5\times\cos(2\pi\times 1.879\times t)=43.89\times\cos(2\pi\times 1.879\times t)\ N/m$

5.1.2.2.3 Calculation of dynamic responses

The acceleration under the vertical load is equal to:

$$A_{cc\max}=\frac{1}{2\xi}\frac{4\times F}{\rho S\pi}=\frac{1}{2\times 0.6/100}\frac{4\times 43.89}{3251\times\pi}=1.43\ m/s^2$$

The maximum acceleration calculated is located within range 3 of the accelerations, in other words at a medium comfort level (between 1 and 2.5 m/sec^2).
For the second mode, the frequency of 2.92 Hz requires us to take the second harmonic into account. But, with a force of one-quarter of that of the first harmonic, and taking into account the above acceleration result, the comfort level obtained is maximum.

5.1.2.3 Class I

We will finally consider class I.

5.1.2.3.1 Calculation of natural modes

The natural modes of the deck were calculated using the Systus program.

For class I, we are interested in a very dense crowd, where the density *d* of the crowd is equal to 1.0 pedestrian/m^2.
The number of pedestrians on the footbridge is: $n_p = 1\times(2\times40)\times3.5 = 280$
The total mass of the pedestrians is: $70\times280 = 19600\,kg$
The linear density of the pedestrians is: $m_p = 19600/80 = 245\,kg/m$
The total linear density is: $\rho S = 3055 + 245 = 3300\,kg/m$
For the first mode, the high-bond and low bond frequencies are equal to:
$f_1 = 1.94\,Hz \quad f_1 = 1.86\,Hz$
For the second mode, the high-bond and low bond frequencies are equal to:
$f_2 = 3.04\,Hz \quad f_2 = 2.92\,Hz$

5.1.2.3.2 Calculation of the dynamic load of the pedestrians

First of all, we calculate the load for the first mode, and a critical damping ratio of 0.6% (mixed deck).
The surface load to be taken into account for the vertical modes is:

$$F_s = d\times(280N)\times\cos 2\pi f_v t\times1.86\times\sqrt{\frac{1}{n}}\times\psi$$

ψ is equal to 1, as the frequency of the first mode, which is 1.86 Hz, is within range 1 of the frequencies (1.7 to 2.1 Hz).
The surface load equals:

$$F_s = 1.0\times280\times\cos(2\pi\times1.86\times t)\times1.85\sqrt{\frac{1}{280}}\times1 = 30.956\cos(2\pi\times1.86\times t)\,N/m^2$$

The linear load is equal to:

$$F = F_s\times l_p = 30.956\times3.5\times\cos(2\pi\times1.86\times t) = 108.35\times\cos(2\pi\times1.86\times t)\,N/m$$

5.1.2.3.3 Calculation of dynamic responses

The acceleration under the vertical load is equal to:

$$A_{cc\max} = \frac{1}{2\xi}\frac{4\times F}{\rho S\pi} = \frac{1}{2\times0.6/100}\frac{4\times108.35}{3300\times\pi} = 3.48\,m/s^2$$

The maximum acceleration calculated is located within range 4 of the accelerations, in other words at an unacceptable comfort level (acceleration > 2.5 m/sec^2).
For the second mode, the frequency of 2.91 Hz requires us to take the second harmonic into account. But, with a force of one-quarter of that of the first harmonic, and taking into account

the above acceleration result, the comfort level obtained is medium (approximately 0.9 m/sec²).

5.1.2.4 Summary

Class	Frequency in Hz	Damping ratio	Acceleration in m/sec^2
III	1.90	0.6	1.16
II	1.88	0.6	1.43
I	1.86	0.6	3.48

Table 5.1

It can be seen that the accelerations are higher than 1 m/sec² for all categories. If the medium comfort level is selected, the project will have to be modified.

In order to reduce the accelerations obtained, the stiffness of the construction must be increased. To do this, the depth of the box girder can be increased (for example by 40 cm).

5.1.2.5 Stiffened steel box girder footbridge

The framework is formed from a steel box girder of a constant depth of 1.40 metres. A 10 cm thick pre-cast reinforced concrete slab bears on this girder.
The moment of inertia of the deck is modified and equals: $I = 0.106\, m^4$
Natural linear density of the deck: $m = 3241\, kg/m$
Young's modulus of the steel: $E = 210 \times 10^9\ N/m^2$

The high bond and low bond frequencies are modified as follows:
$f_1 = 2.57\,Hz \quad f_1 = 2.47\,Hz$ and $f_2 = 4.02\,Hz \quad f_2 = 3.83\,Hz$

For class III, the two natural frequencies are outside the range of frequencies that have to be checked. In class III, the comfort level is therefore automatically maximum.
The accelerations associated with the first mode are, for class I and II:

In class II: $A_{cc\max} = \frac{1}{2\xi}\frac{4 \times F}{\rho S \pi} = \frac{1}{2 \times 0.6/100}\frac{4 \times 10.36}{3437 \times \pi} = 0.32\, m/s^2$

In class I: $A_{cc\max} = \frac{1}{2\xi}\frac{4 \times F}{\rho S \pi} = \frac{1}{2 \times 0.6/100}\frac{4 \times 28.17}{3437 \times \pi} = 0.85\, m/s^2$

For class II, the comfort level is maximum. It is medium for class I, but acceptable even so.
The conclusions on comfort levels, taking the second harmonic of pedestrians walking into account, for the second natural modes, are unchanged.

5.2 - Sensitivity study of typical footbridges.

This section sets out a study of four types of standard footbridges based on actual examples, in which the span of these typical footbridges is varied around the actual span, but remaining within their field of use.

The first paragraph sets out the footbridges studied and the second states their natural frequencies.

5.2.1 - Presentation of the footbridges studied and static pre-sizing

Four main types of footbridges were distinguished: in reinforced concrete, in pre-stressed concrete, mixed steel-concrete box girder and steel lattice.
For each type, rough pre-sizing was carried out, for footbridges of a total length of between 20 and 80 m; the assumptions and the procedure used in each case are given in the following section.
In order to allow a comparison between the various footbridges, the same net width of 3.50 m was taken for all the footbridges. In the same way, the superstructures are comparable, even though they have been adapted to each case.

5.2.1.1 Reinforced concrete footbridge

For reinforced concrete, a distinction is made between two fields of span: small spans (20 to 25 m), for which a rectangular reinforced concrete slab is sufficient, and longer spans (25 to 45 m), for which we will use an H-shaped reinforced footbridge. Only the depth h varies when the span changes, the fixed dimensions are as follows:

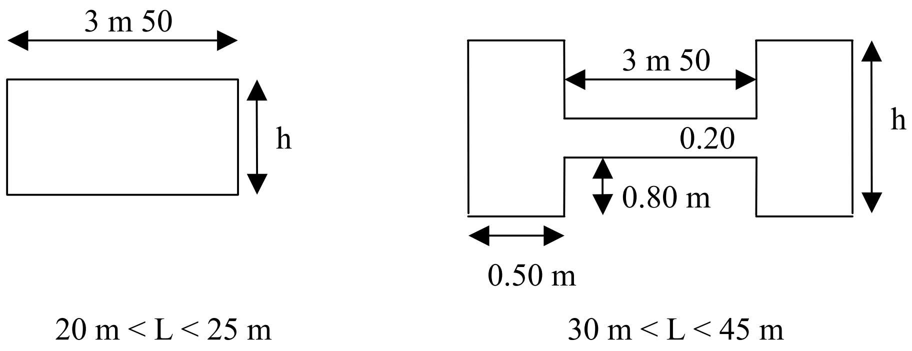


Figure 5.6: Reinforced concrete footbridges

The depth of the footbridge to be used is that which enables the compression limit of the concrete (here 15 MPa) to be guaranteed, and to take the steels, either in the 3.50 m width, for the slab, or in the two 50 cm beams, for the H-shaped footbridge.
The superstructures of this footbridge, together with their weights, are:

- damp-proofing	3.5 x 0.03 x 24 = 2.5 kN /m
- finish	3.5 x 0.04 x 24 = 4 kN /m
- sundries	1 kN / m
i.e.	7.5 kN / m in total.

The role of the balustrade is played by the H-beams themselves. For the reinforced slab, two edge brackets and two balustrades must be added, i.e. 16 kN / m in total.

5.2.1.2 Pre-stressed concrete footbridge

In the same way as for reinforced concrete, a distinction is made between two distinct fields: where L is between 20 m and m, a pre-stressed rectangular slab is used. On the other hand, as soon as L becomes longer than 35 m (up to 50 m), a pre-stressed box girder is preferably to be used. Only the depth h varies when the span changes, the fixed dimensions are as follows:

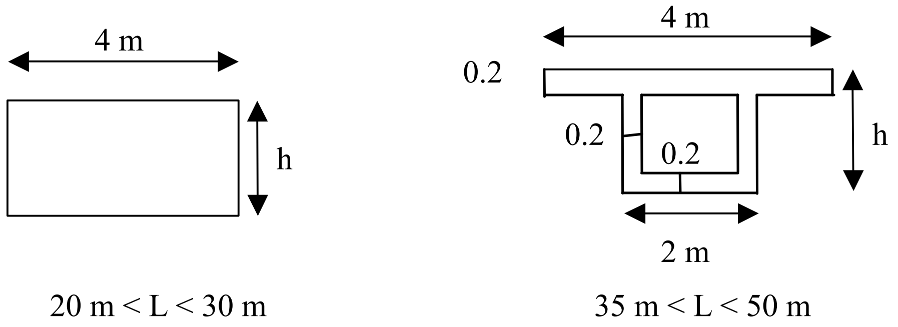


Figure 5.7: Pre-stressed concrete footbridges

The superstructures of this footbridge, together with their weights, are:

- 2 edge brackets 2 x 25 x (0.25 x 0.34) = 4.5 kN / m
- 2 balustrades 2 x 2 = 4 kN / m
- damp-proofing 4 x 0.03 x 24 = 3 kN /m
- finish 3.5 x 0.04 x 24 = 4 kN /m
- sundries 1 kN / m

i.e. 16.5 kN / m in total.

5.2.1.3 Steel or mixed footbridge

This type of footbridge is a steel box girder (welded reconstituted beams) in grade S355 steel. A concrete slab bears on this box girder; if it is connected to it, it becomes a mixed section. If it does not, the load-bearing section is the steel alone, but the thickness of the concrete is still there and adds to the overall weight. Only the depth of the footbridge varies when the span changes, the fixed dimensions are as follows:

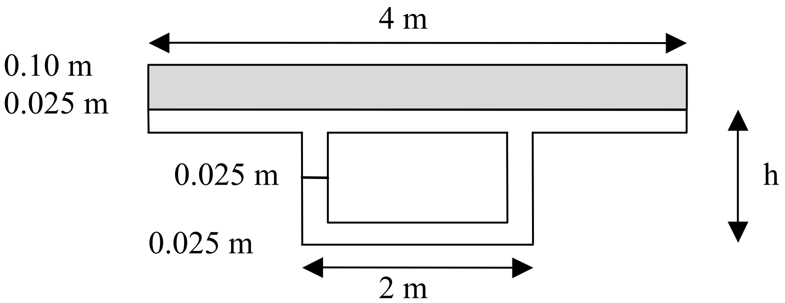


Figure 5.8: Mixed footbridge

We will study this type of footbridge for total spans ranging from 40 to 80 m. For the mixed box girder, the sizing is done on the assumption that the steel section bears its own weight and also that of the wet concrete (assumed placed in a single pour). On the other hand, the mixed section takes the superstructure (with an equivalence factor n = 18) and the static pedestrian loads (factor n = 6). For the steel box girder, the only load-bearing section is the steel girder.
The superstructures of this footbridge, together with their weights, are:

- 2 edge brackets 2 x 25 x (0.25 x 0.34) = 4.5 kN / m
- 2 balustrades 2 x 2 = 4 kN / m
- damp-proofing 4 x 0.03 x 24 = 3 kN /m
- finish 3.5 x 0.04 x 24 = 4 kN /m
- sundries 1 kN / m

i.e. 16.5 kN / m in total.

5.2.1.4 Steel lattice footbridge

The main field of span taken into account for this type of footbridge is 50-80 m. It is assumed that the footbridge is properly triangulated. The values that remain fixed when the span changes are:

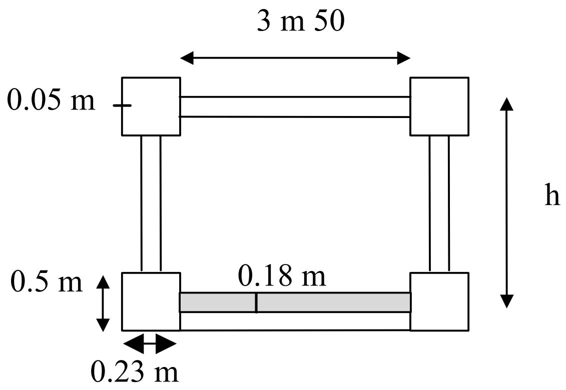


Figure 5.9: Steel lattice footbridge

The superstructures of this footbridge, together with their weights, are:

- 2 balustrades 2 x 2 = 4 kN / m
- damp-proofing 3.5 x 0.03 x 24 = 2.5 kN /m
- finish 3.5 x 0.04 x 24 = 4 kN /m
- sundries 1 kN / m

i.e. 11.5 kN / m in total.

5.2.2 - Natural frequencies

The four previous types of footbridges are considered to be iso-static bays.
For an iso-static bay, the formula giving the natural frequency of the bridge is:

$$f_n = \frac{1}{2\pi}\frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho S}}$$

- f_n is the natural frequency of the mode n;
- L is the length of the bay in m;
- I is the inertia in m^4, vertical or horizontal, depending on what is being sought;
- E is the Young's modulus of the material comprising the structure in N / m²;
- ρS is the linear density of the bridge (self-mass and mass of the superstructures) in kg / m, to which is added the mass of the pedestrians on the footbridge. We have taken 1 pedestrian / m², with 70 kg per pedestrian; i.e. ρS = permanent loads + 3.5 x 70 x 1 in kg / m.

We therefore varied the length of the iso-static bay for each type of footbridge. The depth was determined according to static sizing.
The following tables set out the four natural frequencies: the first two vertical and the first two horizontal.

	mixed box-girder					**steel box-girder**					**lattice girder**				
L		vert. freq. 1	vert. freq. 2	hor. freq. 1	hor. freq. 2		vert. freq. 1	vert. freq. 2	hor. freq. 1	hor. freq. 2		vert. freq. 1	vert. freq. 2	hor. freq. 1	hor. freq. 2

40		1.2	4.6	3.6	14		1.6	6.2	3.1	12					
50		1.1	4.4	2.3	9.3		1.5	6.0	2.0	8.1		1.4	5.6	2.0	8.3
60		1.1	4.2	1.6	6.5		1.4	5.7	1.5	5.9		1.4	5.7	1.4	5.6
70		1.0	4.1	1.2	4.9		1.4	5.5	1.1	4.5		1.4	5.7	1.0	4.1
80		1.0	3.9	1.0	3.8		1.4	5.3	0.9	3.5		1.5	5.8	0.8	3.1

Table 5.2

L		**in RC**				
			vert. freq. 1	vert. freq. 2	hor. freq. 1	hor. freq. 2
slab	20		2.7	11	19	77
RC	25		2.5	10	13	50
H-sect.	25		3.4	13	15	61
	30		3.0	12	11	44
	35		2.7	11	8	33
	40		2.5	10	6	26
	45		2.4	10	5	21

L		**in PC**				
			vert. freq. 1	vert. freq. 2	hor. freq. 1	hor. freq. 2
slab	20		2.0	8.0	15	58
PC	25		1.8	7.1	10	39
	30		1.9	7.7	7.0	28
box girder	30		2.4	9.8	5.2	21
	35		2.3	9.1	3.9	15
	40		2.1	8.5	3.0	12
	45		2.0	8.2	2.4	9.5
	50		2.0	7.9	1.9	7.7

Table 5.3

The following comments may be inferred from these tables:

➢ Only the first mode need be used for these simple types of footbridges, both for vertical and horizontal vibrations;

➢ For vertical vibrations, the first natural frequency of metal footbridges is around 1 – 1.5 Hz, whereas, for concrete footbridges, it is nearer 2 - 3 Hz (more precisely, 2.5 - 3 Hz for reinforced concrete and 2 - 3 Hz for pre-stressed concrete);

➢ Footbridges that are unable to vibrate under pedestrian excitation are rare; these are H-shaped reinforced footbridges of 25 m. All others would appear to be classified as "at risk";

➢ Finally, for lateral vibrations, it would be sufficient to study large-span steel footbridges (for all others, the first horizontal frequency is beyond 2.5 Hz).

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46 avenue
Aristide Briand
BP 100
92225 Bagneux Cedex
France
téléphone :
33 (0)1 46 11 31 31
télécopie :
33 (0)1 46 11 31 69
internet : www.setra.
equipement.gouv.fr

These guidelines round up the state of current knowledge on dynamic behaviour of footbridges under pedestrian loading. An analytical methodology and recommendations are also proposed to guide the designer of a new footbridge when considering the resulting dynamic effects.

The methodology is based on the footbridge classification concept (as a function of traffic level) and on the required comfort level and relies on interpretation of results obtained from tests performed on the Solferino footbridge and on an experimental platform.

These guidelines are aimed at Owners, designers and engineers.

This document covers the following topics:

- a description of the dynamic phenomena specific to footbridges and identification of the parameters which have an impact on the dimensioning of such structures;
- a methodology for the dynamic analysis of footbridges based on a classification according to the traffic level;
- a presentation of the practical methods for calculation of natural frequencies and modes, as well as structural response to loading;
- recommendations for the drafting of design and construction documents.

Supplementary theoretical (reminder of structural dynamics, pedestrian load modelling) and practical (damping systems, examples of recent footbridges, typical calculations) data are also provided in the guideline appendices.

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