Technical Note: Torsional Analysis of Steel Sections

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Torsional analysis of rolled steel sections is generally accomplished with torsional function curves which have been published by the American Institute of Steel Construction in Design Guide No. 9, *Torsional Analysis of Structural Steel Members* (Seaburg and Carter, 1997), which is an update to an earlier Bethlehem Steel publication (Heins and Seaburg, 1963). Such problems are more easily solved with a personal computer than with charts, and this paper presents the equations in a form suitable for programming.

The equations for the angle of twist, θ, for twelve cases of torsional loading and end conditions are given in AISC's Design Guide No. 9, but the successive derivatives of θ (θ', θ'', and θ''') are not given. These derivatives are required to determine the torsional stresses in the following equations:

$$\text{Pure Torsional Shear Stress} = Gt\theta' \qquad (1)$$

$$\text{Warping Normal Stress} = E\,W_{ns}\,\theta'' \qquad (2)$$

$$\text{Warping Shear Stress} = -\frac{ES_{ws}}{t}\theta''' \qquad (3)$$

where

G = shear modulus of elasticity; 11,200 ksi for steel
t = thickness of the element, in.
E = modulus of elasticity; 29,000 ksi for steel
W_{ns} = normalized warping function at a point s on the cross section, in.^{2}
S_{ws} = warping statical moment at a point s on the cross section, in.^{4}

The reader is referred to the AISC Design Guide No. 9 (Seaburg and Carter, 1997) for a general discussion of torsional stresses, end conditions and combining torsional stresses with bending and shear stresses.

The user is encouraged to program the pinned and fixed end conditions on the same output in order to compare the trade-off between rotational stiffness and warping normal stress.

The equations for θ, θ', θ'', and θ''' for the twelve commonly encountered loading and end conditions presented in the AISC Design Guide No. 9 are given in the Appendix of this paper. The case numbers are consistent with those given in the design guide. These equations contain no dimensional factors and may be used with any consistent set of units. Each set of equations has been tested against a solution using the curves from the AISC Design Guide (Seaburg and Carter, 1997). Although the additional precision afforded by using the equations (instead of the curves) is of doubtful value, the ease and speed of computation is immensely helpful.

REFERENCES

Seaburg, P.A. and Carter, C.J. (1997), *Torsional Analysis of Structural Steel Members*, Steel Design Guide Series No. 9, AISC, Chicago, IL.

Heins, C.P. and Seaburg, P.A. (1963), *Torsion Analysis of Rolled Steel Sections*, Bethlehem Steel Company Steel Design File, 1963-B.

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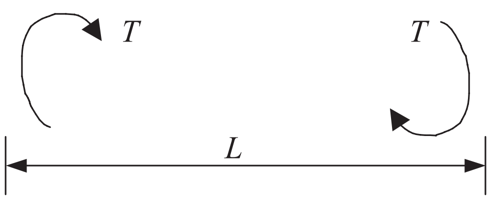
APPENDIX

The following variables and constants are used throughout the appendix:

a = $\sqrt{\dfrac{EC_w}{GJ}}$
G = shear modulus of elasticity; 11,200 ksi for steel
E = modulus of elasticity; 29,000 ksi for steel
C_w = warping constant, in.^{2}
J = torsional constant, in.^{4}
T = concentrated torque, kip-in.
t = running torque load, kip-in./ft
L = span length, in.
z = length from left support to cross-section analyzed, in.

Case 1—Concentrated Torques with Free Ends

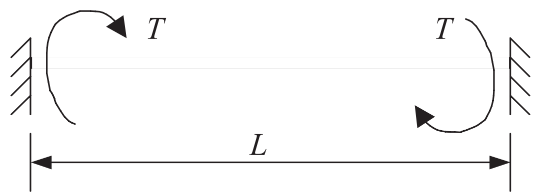


$$\theta = \frac{Tz}{GJ}$$

$$\theta' = \frac{T}{GJ}$$

$$\theta'' = \theta''' = 0$$

Case 2—Concentrated Torques with Fixed Ends



$$\theta = c_1\left(c_2 \cosh\frac{z}{a} - c_2 + \frac{z}{a} - \sinh\frac{z}{a}\right)$$

$$\theta' = \frac{c_1}{a}\left(1.0 - \cosh\frac{z}{a} + c_2 \sinh\frac{z}{a}\right)$$

$$\theta'' = \frac{c_1}{a^2}\left(c_2 \cosh\frac{z}{a} - \sinh\frac{z}{a}\right)$$

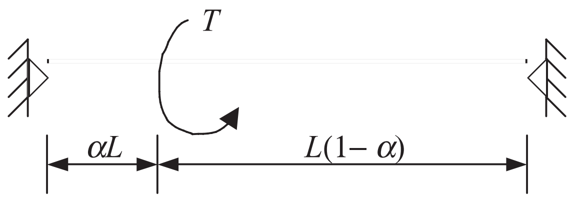
$$\theta''' = \frac{c_1}{a^3}\left(-\cosh\frac{z}{a} + c_2 \sinh\frac{z}{a}\right)$$

where

$$c_1 = \frac{Ta}{GJ}$$

$$c_2 = \tanh\frac{L}{2a}$$

Case 3—Concentrated Torque with Pinned Ends



$0 \le z \le \alpha L$

$$\theta = c_1\left[(1.0-\alpha)z + c_2 a \sinh\frac{z}{a}\right]$$

$$\theta' = c_1\left[(1.0-\alpha) + c_2\cosh\frac{z}{a}\right]$$

$$\theta'' = \frac{c_1 c_2}{a}\sinh\frac{z}{a}$$

$$\theta''' = \frac{c_1 c_2}{a^2}\cosh\frac{z}{a}$$

$\alpha L < z \le L$

$$\theta = c_1\left\{(L-z)\alpha + a\left[\frac{\sinh\frac{\alpha L}{a}}{\tanh\frac{L}{a}}\sinh\frac{z}{a} - \sinh\frac{\alpha L}{a}\cosh\frac{z}{a}\right]\right\}$$

$$\theta' = c_1\left(\frac{\sinh\frac{\alpha L}{a}}{\tanh\frac{L}{a}}\cosh\frac{z}{a} - \sinh\frac{\alpha L}{a}\sinh\frac{z}{a} - \alpha\right)$$

$$\theta'' = \frac{c_1}{a}\left(\frac{\sinh\frac{\alpha L}{a}}{\tanh\frac{L}{a}}\sinh\frac{z}{a} - \sinh\frac{\alpha L}{a}\cosh\frac{z}{a}\right)$$

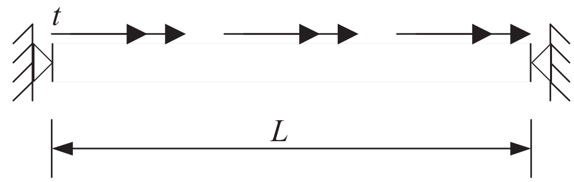
$$\theta''' = \frac{c_1}{a^2}\left(\frac{\sinh\frac{\alpha L}{a}}{\tanh\frac{L}{a}}\cosh\frac{z}{a} - \sinh\frac{\alpha L}{a}\sinh\frac{z}{a}\right)$$

where

$$c_1 = \frac{T}{GJ}$$

$$c_2 = \frac{\sinh\frac{\alpha L}{a}}{\tanh\frac{L}{a}} - \cosh\frac{\alpha L}{a}$$

Case 4—Uniformly Distributed Torque with Pinned Ends



$$\theta = c_1 a^2\left[\frac{L}{2a^2}\left(z - \frac{z^2}{L}\right) + \cosh\frac{z}{a} - \tanh\frac{L}{2a}\sinh\frac{z}{a} - 1.0\right]$$

$$\theta' = c_1 a\left[\frac{L}{2a}\left(1.0 - \frac{2z}{L}\right) + \sinh\frac{z}{a} - \tanh\frac{L}{2a}\cosh\frac{z}{a}\right]$$

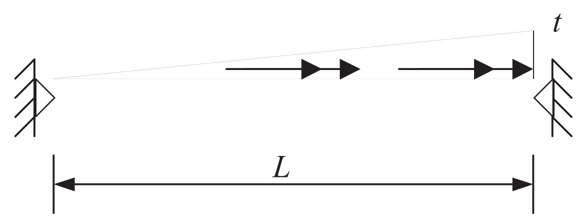
$$\theta'' = c_1\left(\cosh\frac{z}{a} - \tanh\frac{L}{2a}\sinh\frac{z}{a} - 1.0\right)$$

$$\theta''' = \frac{c_1}{a}\left(\sinh\frac{z}{a} - \tanh\frac{L}{2a}\cosh\frac{z}{a}\right)$$

where

$$c_1 = \frac{t}{GJ}$$

Case 5—Linearly Varying Torque with Pinned Ends



$$\theta = c_1 L\left(\frac{z}{6} - \frac{za^2}{L^2} + \frac{a^2}{L}\frac{\sinh\frac{z}{a}}{\sinh\frac{L}{a}} - \frac{z^3}{6L^2}\right)$$

$$\theta' = c_1 L\left(\frac{1}{6} - \frac{a^2}{L^2} + \frac{a}{L}\frac{\cosh\frac{z}{a}}{\sinh\frac{L}{a}} - \frac{z^2}{2L^2}\right)$$

$$\theta'' = c_1\left(\frac{\sinh\frac{z}{a}}{\sinh\frac{L}{a}} - \frac{z}{L}\right)$$

$$\theta''' = c_1\left(\frac{\cosh\frac{z}{a}}{a\sinh\frac{L}{a}} - \frac{1}{L}\right)$$

where

$$c_1 = \frac{t}{GJ}$$

Case 6—Concentrated Torque with Fixed Ends

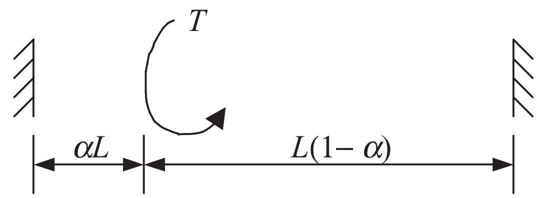
$0 \le z \le \alpha L$

$$\theta = c_1 a\left[c_2\left(\cosh\frac{z}{a} - 1.0\right) - \sinh\frac{z}{a} + \frac{z}{a}\right]$$

$$\theta' = c_1\left(c_2 \sinh\frac{z}{a} - \cosh\frac{z}{a} + 1.0\right)$$

$$\theta'' = \frac{c_1}{a}\left(c_2 \cosh\frac{z}{a} - \sinh\frac{z}{a}\right)$$

$$\theta''' = \frac{c_1}{a^2}\left(c_2 \sinh\frac{z}{a} - \cosh\frac{z}{a}\right)$$



$\alpha L < z \le L$

$$\theta = Hc_1 a\left(c_3 + c_4 \cosh\frac{z}{a} + c_5 \sinh\frac{z}{a} - \frac{z}{a}\right)$$

$$\theta' = Hc_1\left(c_4 \sinh\frac{z}{a} + c_5 \cosh\frac{z}{a} - 1.0\right)$$

$$\theta'' = \frac{Hc_1}{a}\left(c_4 \cosh\frac{z}{a} + c_5 \sinh\frac{z}{a}\right)$$

$$\theta''' = \frac{Hc_1}{a^2}\left(c_4 \sinh\frac{z}{a} + c_5 \cosh\frac{z}{a}\right)$$

where

$$H = \frac{\left[\dfrac{\left(1.0 - \cosh\frac{\alpha L}{a}\right)}{\tanh\frac{L}{a}} + \dfrac{\left(\cosh\frac{\alpha L}{a} - 1.0\right)}{\sinh\frac{L}{a}} + \sinh\frac{\alpha L}{a} - \frac{\alpha L}{a}\right]}{\left[\dfrac{\left(\cosh\frac{L}{a} + \cosh\frac{\alpha L}{a}\cosh\frac{L}{a} - \cosh\frac{\alpha L}{a} - 1.0\right)}{\sinh\frac{L}{a}} + \frac{L}{a}(\alpha - 1.0) - \sinh\frac{\alpha L}{a}\right]}$$

$$c_1 = \frac{T}{(H+1)GJ}$$

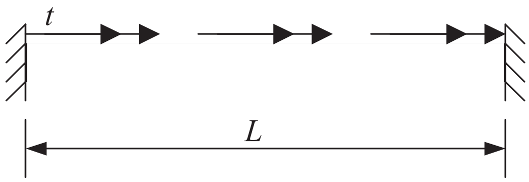
$$c_2 = H\left(\frac{1}{\sinh\frac{L}{a}} + \sinh\frac{\alpha L}{a} - \frac{\cosh\frac{\alpha L}{a}}{\tanh\frac{L}{a}}\right) + \sinh\frac{\alpha L}{a} - \frac{\cosh\frac{\alpha L}{a}}{\tanh\frac{L}{a}} + \frac{1}{\tanh\frac{L}{a}}$$

$$c_3 = \frac{\left(\cosh\frac{\alpha L}{a} - 1.0\right)}{H\sinh\frac{L}{a}} + \frac{\left(\cosh\frac{\alpha L}{a} - \cosh\frac{L}{a} + \frac{L}{a}\sinh\frac{L}{a}\right)}{\sinh\frac{L}{a}}$$

$$c_4 = \frac{\left(1.0 - \cosh\frac{\alpha L}{a}\right)}{H\tanh\frac{L}{a}} + \frac{\left(1.0 - \cosh\frac{\alpha L}{a}\cosh\frac{L}{a}\right)}{\sinh\frac{L}{a}}$$

$$c_5 = \frac{\left(\cosh\frac{\alpha L}{a} - 1.0\right)}{H} + \cosh\frac{\alpha L}{a}$$

Case 7—Uniformly Distributed Torque with Fixed Ends



$$\theta = c_1 a\left[c_2\left(\cosh\frac{z}{a} - 1.0\right) + \frac{z}{a}\left(1 - \frac{z}{L}\right) - \sinh\frac{z}{a}\right]$$

$$\theta' = c_1\left(c_2 \sinh\frac{z}{a} + 1.0 - \frac{2z}{L} - \cosh\frac{z}{a}\right)$$

$$\theta'' = \frac{c_1}{a}\left(c_2 \cosh\frac{z}{a} - \frac{2a}{L} - \sinh\frac{z}{a}\right)$$

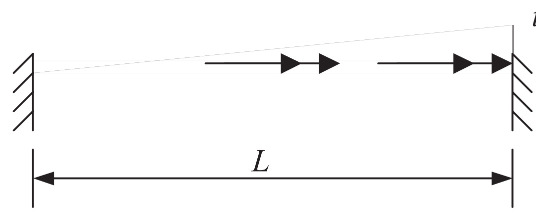
$$\theta''' = \frac{c_1}{a^2}\left(c_2 \sinh\frac{z}{a} - \cosh\frac{z}{a}\right)$$

where

$$c_1 = \frac{tL}{2GJ}$$

$$c_2 = \frac{1 + \cosh\frac{L}{a}}{\sinh\frac{L}{a}}$$

Case 8—Linearly Varying Torque with Fixed Ends



$$\theta = c_1\left[c_2\left(\cosh\frac{z}{a} - 1.0\right) + S\left(\sinh\frac{z}{a} - \frac{z}{a}\right) - \frac{z^3}{6L^3}\right]$$

$$\theta' = \frac{c_1}{a}\left(c_2 \sinh\frac{z}{a} + S\cosh\frac{z}{a} - S - \frac{z^2 a}{2L^3}\right)$$

$$\theta'' = \frac{c_1}{a^2}\left(c_2 \cosh\frac{z}{a} + S\sinh\frac{z}{a} - \frac{za^2}{L^3}\right)$$

$$\theta'' = \frac{c_1}{a^3}\left(c_2 \sinh\frac{z}{a} + S\cosh\frac{z}{a} - \frac{a^3}{L^3}\right)$$

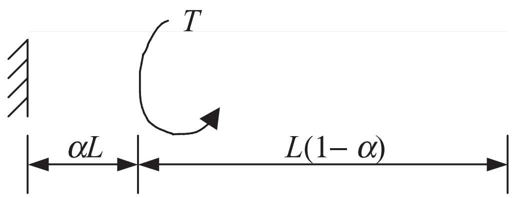
where

$$c_1 = \frac{tL^2}{GJ}$$

$$c_2 = \frac{a}{2L\sinh\frac{L}{a}} - S\tanh\frac{L}{2a}$$

$$S = \left[\frac{\frac{a}{2L}\left(\cosh\frac{L}{a} - 1.0\right) - \frac{\sinh\frac{L}{a}}{6.0}}{\frac{L}{a}\sinh\frac{L}{a} + 2.0 - 2\cosh\frac{L}{a}}\right]$$

Case 9—Concentrated Torque with Fixed and Free End



$0 \le z \le \alpha L$

$$\theta = c_1 a\left[c_2\left(\cosh\frac{z}{a} - 1.0\right) - \sinh\frac{z}{a} + \frac{z}{a}\right]$$

$$\theta' = c_1\left(c_2\sinh\frac{z}{a} - \cosh\frac{z}{a} + 1.0\right)$$

$$\theta'' = \frac{c_1}{a}\left(c_2\cosh\frac{z}{a} - \sinh\frac{z}{a}\right)$$

$$\theta''' = \frac{c_1}{a^2}\left(c_2\sinh\frac{z}{a} - \cosh\frac{z}{a}\right)$$

$\alpha L < z \le L$

$$\theta = c_1 a\left(c_3 - c_4\tanh\frac{L}{a}\cosh\frac{z}{a} + c_4\sinh\frac{z}{a} + \frac{\alpha L}{a}\right)$$

$$\theta' = c_1 c_4\left(-\tanh\frac{L}{a}\sinh\frac{z}{a} + \cosh\frac{z}{a}\right)$$

$$\theta'' = \frac{c_1 c_4}{a}\left(-\tanh\frac{L}{a}\cosh\frac{z}{a} + \sinh\frac{z}{a}\right)$$

$$\theta''' = \frac{c_1 c_4}{a^2}\left(-\tanh\frac{L}{a}\sinh\frac{z}{a} + \cosh\frac{z}{a}\right)$$

where

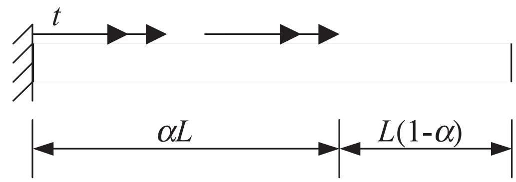
$$c_1 = \frac{T}{GJ}$$

$$c_2 = \sinh\frac{\alpha L}{a} - \tanh\frac{L}{a}\cosh\frac{\alpha L}{a} + \tanh\frac{L}{a}$$

$$c_3 = \tanh\frac{L}{a}\cosh\frac{\alpha L}{a} - \tanh\frac{L}{a} - \sinh\frac{\alpha L}{a}$$

$$c_4 = \cosh\frac{\alpha L}{a} - 1.0$$

Case 10—Partially Uniformly Distributed Torque with Fixed and Free End



$0 \le z \le \alpha L$

$$\theta = c_1 a\left[c_2 a\left(\cosh\frac{z}{a} - 1.0\right) - \alpha L\sinh\frac{z}{a} + z\left(\frac{\alpha L}{a} - \frac{z}{2a}\right)\right]$$

$$\theta' = c_1\left(c_2 a\sinh\frac{z}{a} - \alpha L\cosh\frac{z}{a} + \alpha L - z\right)$$

$$\theta'' = \frac{c_1}{a}\left(c_2 a\cosh\frac{z}{a} - \alpha L\sinh\frac{z}{a} - a\right)$$

$$\theta''' = \frac{c_1}{a^2}\left(c_2 a\sinh\frac{z}{a} - \alpha L\cosh\frac{z}{a}\right)$$

$\alpha L < z \le L$

$$\theta = c_1 a^2\left(c_3 - c_4\cosh\frac{z}{a} + c_5\sinh\frac{z}{a}\right)$$

$$\theta' = c_1 a\left(-c_4\sinh\frac{z}{a} + c_5\cosh\frac{z}{a}\right)$$

$$\theta'' = c_1\left(-c_4\cosh\frac{z}{a} + c_5\sinh\frac{z}{a}\right)$$

$$\theta''' = \frac{c_1}{a}\left(-c_4\sinh\frac{z}{a} + c_5\cosh\frac{z}{a}\right)$$

where

$$c_1 = \frac{t}{GJ}$$

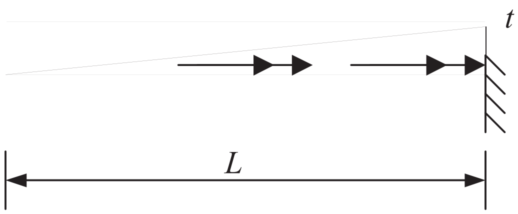
$$c_2 = \tanh\frac{L}{a}\left(\frac{\alpha L}{a} - \sinh\frac{\alpha L}{a}\right) + \cosh\frac{\alpha L}{a}$$

$$c_3 = \tanh\frac{L}{a}\sinh\frac{\alpha L}{a} - \cosh\frac{\alpha L}{a} - \frac{\alpha L}{a}\tanh\frac{L}{a} + 1.0 + \frac{\alpha^2 L^2}{2a^2}$$

$$c_4 = \left(\sinh\frac{\alpha L}{a} - \frac{\alpha L}{a}\right)\tanh\frac{L}{a}$$

$$c_5 = \sinh\frac{\alpha L}{a} - \frac{\alpha L}{a}$$

Case 11—Linearly Varying Torque with Free and Fixed End



$$\theta = c_1\left[\begin{array}{l} 1.0 - \dfrac{5L^2}{6a^2} - \left(\dfrac{a}{L} - \dfrac{L}{2a}\right)\tanh\dfrac{L}{a} + z\left(-\dfrac{1}{L} + \dfrac{L}{a^2}\right) \\ + \left(\dfrac{a}{L} - \dfrac{L}{2a}\right)\left(\dfrac{\sinh\dfrac{z}{a}}{\cosh\dfrac{L}{a}}\right) - \dfrac{z^3}{6a^2 L} \end{array}\right]$$

$$\theta' = \frac{c_1}{a}\left[-\frac{a}{L} + \frac{L}{a} + \left(\frac{a}{L} - \frac{L}{2a}\right)\left(\frac{\cosh\frac{z}{a}}{\cosh\frac{L}{a}}\right) - \frac{z^2}{2aL}\right]$$

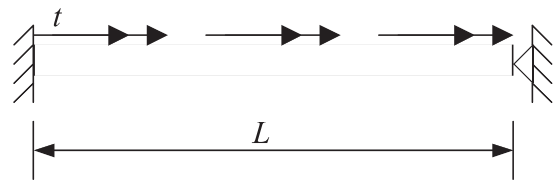
$$\theta'' = \frac{c_1}{a^2}\left[\left(\frac{a}{L} + \frac{L}{2a}\right)\left(\frac{\sinh\frac{z}{a}}{\cosh\frac{L}{a}}\right) - \frac{z}{L}\right]$$

$$\theta''' = \frac{c_1}{a^3}\left[\left(\frac{a}{L} - \frac{L}{2a}\right)\left(\frac{\cosh\frac{z}{a}}{\cosh\frac{L}{a}}\right) - \frac{a}{L}\right]$$

where

$$c_1 = \frac{ta^2}{GJ}$$

Case 12—Uniformly Distributed Torque with Fixed and Pinned Ends



$$\theta = c_1\left[\begin{array}{l} H\left(\tanh\dfrac{L}{a} - \dfrac{z}{a} - \tanh\dfrac{L}{a}\cosh\dfrac{z}{a} + \sinh\dfrac{z}{a}\right) \\ + \dfrac{\cosh\dfrac{z}{a}}{\cosh\dfrac{L}{a}} - \dfrac{1}{\cosh\dfrac{L}{a}} - \dfrac{z^2}{2a^2} \end{array}\right]$$

$$\theta' = \frac{c_1}{a}\left[H\left(-1.0 - \tanh\frac{L}{a}\sinh\frac{z}{a} + \cosh\frac{z}{a}\right) + \frac{\sinh\frac{z}{a}}{\cosh\frac{L}{a}} - \frac{z}{a}\right]$$

$$\theta'' = \frac{c_1}{a^2}\left[H\left(-\tanh\frac{L}{a}\cosh\frac{z}{a} + \sinh\frac{z}{a}\right) + \frac{\cosh\frac{z}{a}}{\cosh\frac{L}{a}} - 1.0\right]$$

$$\theta''' = \frac{c_1}{a^3}\left[H\left(-\tanh\frac{L}{a}\sinh\frac{z}{a} + \cosh\frac{z}{a}\right) + \frac{\sinh\frac{z}{a}}{\cosh\frac{L}{a}}\right]$$

where

$$c_1 = \frac{ta^2}{GJ}$$

$$H = \left(\frac{L^2}{2a^2} - 1.0 + \frac{1}{\cosh\frac{L}{a}}\right)\left(\frac{1}{\tanh\frac{L}{a} - \frac{L}{a}}\right)$$