Sequence Impedances of Transmission Lines

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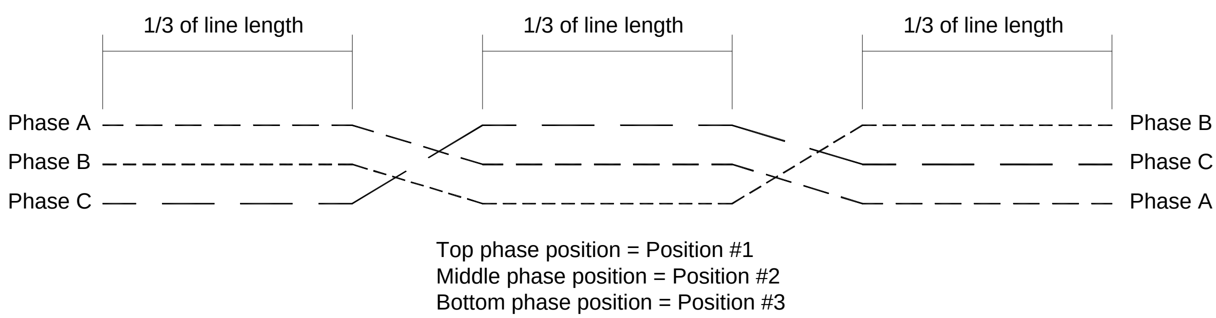
In order to analyze unbalanced conditions on transmission lines, we need to apply the method of symmetrical components, as described by Charles Fortescue in his monumental 1918 AIEE paper[1]. To do so, we first need to express the impedance of a transmission line as positive-, negative-, and zero-sequence components. The determination of sequence impedances for transmission lines is perhaps best explained by Edith Clarke in her classic 1950 text[2] . A brief summary follows.

Positive and Negative Sequence Impedances. A transmission line is a *passive* and *bilateral* device. By passive, we mean there are no voltage or current sources present in the equivalent model of a transmission line. Bilateral means the line behaves the same way regardless of the direction of the current. *Note that although a single transmission line is bilateral, an interconnected transmission network is NOT bilateral, due to the dispersion of active components (generators) throughout the network.* Because of a transmission line's passive and bilateral properties, the phase sequence of the applied voltage makes no difference, as a-b-c (positive-sequence) voltages produce the same voltage drops as a-c-b (negative-sequence) voltages. This means that the positive- and negative-sequence impedances of a transmission line are identical, provided that the line is *transposed*. Transposition means physically exchanging the position of each phase conductor along the length of the line such that conductor #1 occupies: position #1 for 1/3 of the line length, position #2 for 1/3 of the line length, and position #3 for 1/3 of the line length. Conductors #2 and #3 are rotated similarly. The figure below is a plan view (view from above) of a completely transposed transmission line with its phases oriented horizontally (such as with an H-frame structure).



It should be noted that because of the design and construction complications introduced by transposing (usually three special – as well as large and unsightly – structures and additional right-of-way width are needed at each transposition point), most transmission lines built today are NOT transposed. The effects of not transposing a transmission line will be discussed later.

For a completely transposed transmission line,

$$Z_1 = Z_2 = R_1 + j\,X_1 \quad (\Omega/\text{phase})$$

where

R_1 = line resistance to positive sequence currents
X_1 = line inductive reactance to positive sequence currents.

R is a function of both conductor temperature and frequency. X depends on the inductance of the line, and can be expressed as

$$X = 2.020 \times 10^{-3}\ f\ \ln\left(\frac{D_m}{D_s}\right)\ (\Omega\,/\,\text{mile})$$

where

f = frequency in hertz
D_m = mutual geometric mean distance (GMD)
D_s = self geometric mean distance, or geometric mean radius (GMR)

Skin effect. The derivation of R and X above assume uniform current density through the cross section of the conductor. Uniform current density occurs only when the frequency is zero (direct current). As the frequency increases from zero, the current density increases near the conductor surface and decreases as the center of the conductor is approached. This phenomena, known as *skin effect*, reduces the internal flux linkages, and lowers the internal inductance compared to the DC state. It also increases the resistance.

A factor α can be derived for both resistance and inductance, expressing the ratio of R or L at a given frequency to that at DC conditions. This ratio is actually the solution to a differential equation whose closed-form solution is expressed as Bessel functions. The expressions for α_R and α_L are shown below:

$$\alpha_R = \frac{R_f}{R_{DC}} = \frac{mr}{2}\left\{\frac{\text{ber}(mr)\ \text{bei}'(mr) - \text{bei}(mr)\ \text{ber}'(mr)}{[\text{ber}'(mr)]^2 + [\text{bei}'(mr)]^2}\right\}$$

$$\alpha_L = \frac{L_f}{L_{DC}} = \frac{4}{mr}\left\{\frac{\text{ber}(mr)\ \text{ber}'(mr) + \text{bei}(mr)\ \text{bei}'(mr)}{[\text{ber}'(mr)]^2 + [\text{bei}'(mr)]^2}\right\}$$

where

$$m = \sqrt{\omega\mu\sigma}$$

r = conductor radius

Since $\mu = \mu_r \mu_0$, we can write

$$mr = 0.0636 \sqrt{\frac{\mu_r f}{R_{DC}}}$$

The Bessel functions and their first derivatives can be approximated by using series expansions:

$$\mathrm{ber}(mr) = 1 - \frac{(mr)^4}{2^2 \cdot 4^2} + \frac{(mr)^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \ldots$$

$$\mathrm{bei}(mr) = \frac{(mr)^2}{2^2} - \frac{(mr)^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

and

$$\mathrm{ber}'(mr) = \frac{d}{d(mr)} \mathrm{ber}(mr) = \frac{1}{m} \frac{d}{dr} \mathrm{ber}(mr)$$

$$\mathrm{bei}'(mr) = \frac{d}{d(mr)} \mathrm{bei}(mr) = \frac{1}{m} \frac{d}{dr} \mathrm{bei}(mr)$$

Mutual coupling. When any wire making up a transmission line carries a non-zero current, the magnetic flux produced by that current will link with the other wires making up the line, inducing a voltage in the other wires. The flux produced by a current in wire 1 that links wire 2 can be calculated as follows:

$$\lambda_{21} = \oint_2 A \cdot ds_2 \quad (\mathrm{Wb-turns})$$

where

λ_{21} = magnetic flux linkage of wire 2 due to current in wire 1

A = magnetic vector potential = $\frac{\mu_0 I_1}{4\pi} \oint_1 \frac{1}{r}\, ds_1$ (Wb/m)

Magnetic flux linkages produce a mutual inductance between the conductors of the transmission line. Under balanced conditions, no mutual inductance exists, since the sum of the line currents equals zero. But with zero-sequence current, a strong coupling effect exists since the zero-sequence current in each phase is additive. The coupling effect can be represented by the mutual inductance M (in henrys), where

$$M = \frac{\lambda_{21}}{I_1} \ \text{(H)}$$

Effects of transposition. After the self- and mutual-impedances of a transmission line have been determined, the voltage drop along the line can be expressed by writing Ohm's Law in matrix form. Referring to the figure on the first page showing a fully transposed line, the matrix equation for the first section of line (when Phase A occupies Position 1, Phase B occupies Position 2, and Phase C occupies Position 3), is

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

For the middle section of line (when Phase A occupies Position 2, Phase B occupies Position 3, and Phase C occupies Position 1), the matrix equation becomes

$$\begin{bmatrix} V_3 \\ V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{31} & Z_{32} & Z_{33} \\ Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

This change represents a linear transformation of the impedance matrix. Applying the same transformation for the third section of line (when Phase A occupies Position 3, Phase B occupies Position 1, and Phase C occupies Position 2), the equation becomes

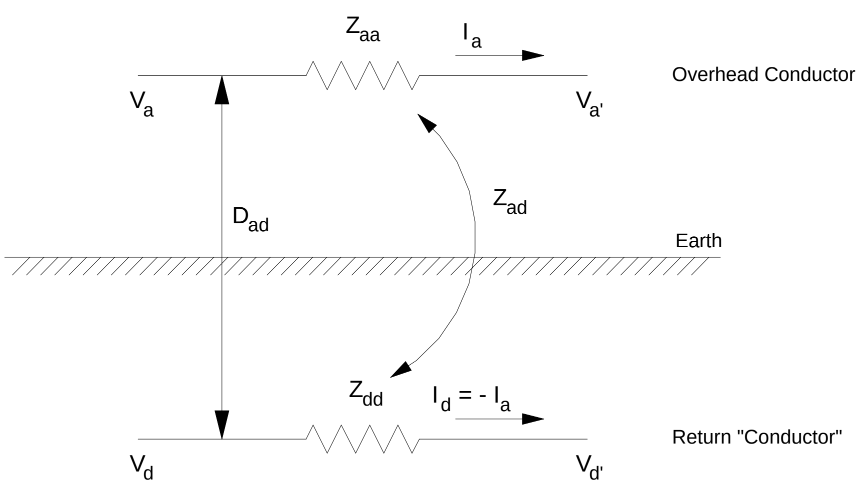
$$\begin{bmatrix} V_2 \\ V_3 \\ V_1 \end{bmatrix} = \begin{bmatrix} Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \\ Z_{11} & Z_{12} & Z_{13} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The impedance of the total line is the sum of the impedance matrices for each of the three sections. When the line is completely transposed, the off-diagonal elements of the impedance matrix are very nearly balanced. With non-transposed lines, the imaginary portion (reactance) of the off-diagonal terms vary greatly, leading to unbalanced voltages between the three phases.

Carson's Equations. Three-phase transmission lines have only three current-carrying conductors. The earth, or a parallel combination of the earth and overhead ground wires (static wires), can carry zero-sequence current, thus changing the zero-sequence impedance of the line. A landmark paper was published by John Carson in 1923[3] describing the impedance of an overhead conductor with earth return.

The earth is considered to have a uniform resistivity and to be of infinite extent. Carson considered the fictitious return conductor to be below the surface of the earth. The return current spreads out over a very large area, seeking the lowest resistance path back to the source. But the return "conductor" can be thought of as being a single conductor with a GMR of 1 foot (or meter) located at a distance D_{ad} (feet or meters) below the overhead conductor. D_{ad} depends on the resistivity (ρ) of the earth, and can be adjusted such that the calculated inductance matches the measured inductance. Empirical values of ρ and D_{ad} are tabulated below:

Return Earth Condition	Resistivity (ρ) in Ω-m	D_{ad} in meters
Sea water	0.01 – 1.0	1.6 – 5.1
Swampy ground	10 – 100	9.1 – 16.1
"Average" damp earth	100	16.1
Dry earth	1000	28.6
Pure slate	10^7	285
Sandstone	10^9	905



A single-phase representation of Carson's line can be shown as follows:

We can write the following equation for Carson's line:

$$\begin{bmatrix} V_{aa'} \\ V_{dd'} \end{bmatrix} = \begin{bmatrix} z_{aa} & z_{ad} \\ z_{ad} & z_{dd} \end{bmatrix} \cdot \begin{bmatrix} I_a \\ -I_a \end{bmatrix}$$

The self impedances z_{aa} and z_{dd} and the mutual impedance z_{ad} can be expressed as

$$z_{aa} = (r_a + r_d) + j\,\omega\,(2\times10^{-7})\ln\left(\frac{D_{ad}^2 \div D_{sd}}{D_{sa}}\right)\ (\Omega/m)$$

$$z_{dd} = r_d + j\ \omega\ (2 \times 10^{-7}) \left[\ln\left(\frac{2\ s}{D_{sd}} \right) - 1 \right] \quad (\Omega / m)$$

and

$$z_{ad} = j\ \omega\ (2 \times 10^{-7}) \left[\ln\left(\frac{2\ s}{D_{ad}} \right) - 1 \right] \quad (\Omega / m)$$

where

D_{ad} = distance between conductor and return “conductor” in meters
$D_{sd} \equiv$ 1 meter (unit length to correct units)
D_{sa} = GMR of conductor in meters
D_{sd} = GMR of return “conductor” in meters
s = length of conductor in meters

The following approximation for the equivalent depth of return can be helpful:

$$D_{ad}^2 \div D_{sd} \approx 2160 \sqrt{\frac{\rho}{f}} \quad (\text{ft})$$

Bibliography

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