ELASTIC BUCKLING STRENGTH OF BRACED BEAMS

by

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Notation

The following symbols are used:

$\bar{a}$ = height of point of application of load above shear centre;
BF = bottom flange;
$\bar{b}$ = height of lateral brace above shear centre;
C = subscript referring to critical segment;
c = ratio of buckling load of braced beam and similar unbraced beam;
E = Young's modulus of elasticity;
G = shear modulus of elasticity;
G_A, G_B = restraint parameters;
h = distance between flange centroids;
I_y = minor axis second moment of area;
I_ω = warping section constant;
J = torsion section constant;
K = beam parameter;
k = effective length factor;
k_s = brace stiffness;
L = length of beam;
M, M_E = elastic buckling moment;
M_b = allowable moment;
M_o = elastic buckling moment of beam under uniform moment ($\beta = -1$);
M_F = elastic buckling moment of substructure;
M_p = plastic moment;
M_Y = first yield moment;
m = moment modification factor;
n = stiffness coefficient, commonly 2, 3, 4;
R = subscript referring to restraining segment;
r_y = minor axis radius of gyration;
SC = shear centre;
TF = top flange;
u = lateral deflection of shear centre;
x, y = major and minor principal axes;
z = centroidal axis;
α = location of brace;
β = moment gradient or ratio of major axis end moment;
λ = load factor
ϕ = angle of twist.

1. Introduction

Beams of open cross-section such as I-beams are usually loaded so as to bend about the major axis. The design of such beams requires consideration of bending strength and deflection. Bending strength may be limited by the plastic moment capacity, M_p, for heavily braced beams, by local buckling loads for thin flanged fabricated beams or by the flexural-torsional or lateral buckling load for beams not so heavily braced. Beam failure by flexural-torsional buckling is shown in Fig.1.

1.1 CLASSIFICATION OF BUCKLING FAILURES

A typical classification of buckling failure types is presented in Fig. 2 which shows possible load-deflection curves for a beam with a mid-point load. Beam slenderness decreases from A to D. Curve A is for a slender beam which buckles while fully elastic. Curve B is for a beam of intermediate slenderness which is partially yielded at failure. Geometrical and material imperfections influence the yielding pattern and hence the capacity. Curve C applies to a beam with a fully plastic zone but without the lateral support necessary for plastic hinge action. Most practical beams would exhibit type B or C failure under increasing loads. Beams of adequate rotational capacity under fully plastic conditions are described as compact beams. Curve D is intended to illustrate this behaviour. Compact beams may exhibit lateral buckling deformations before eventual collapse following local buckling.

1.2 INFLUENCE OF IMPERFECTIONS

The phenomenon of beam buckling has been studied theoretically and experimentally. Most theoretical analyses assume an initially straight beam and seek the load at which the beam enters a state of neutral

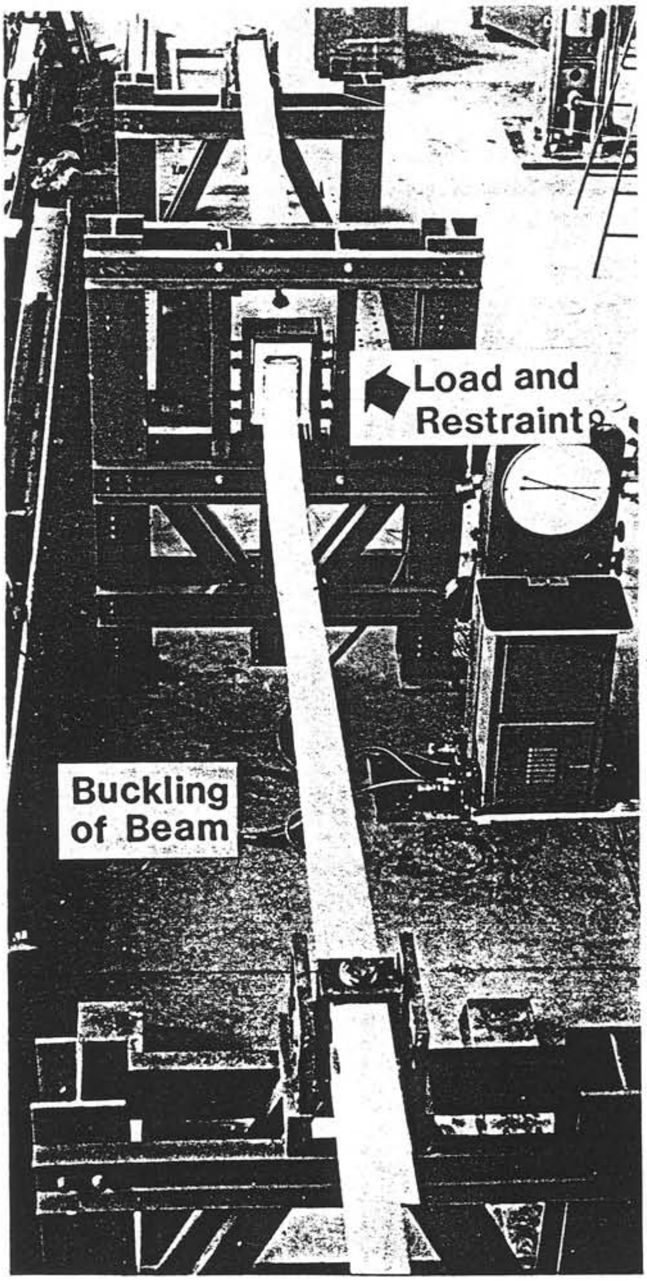


Fig. 1 — Lateral Buckling of I-Beam.

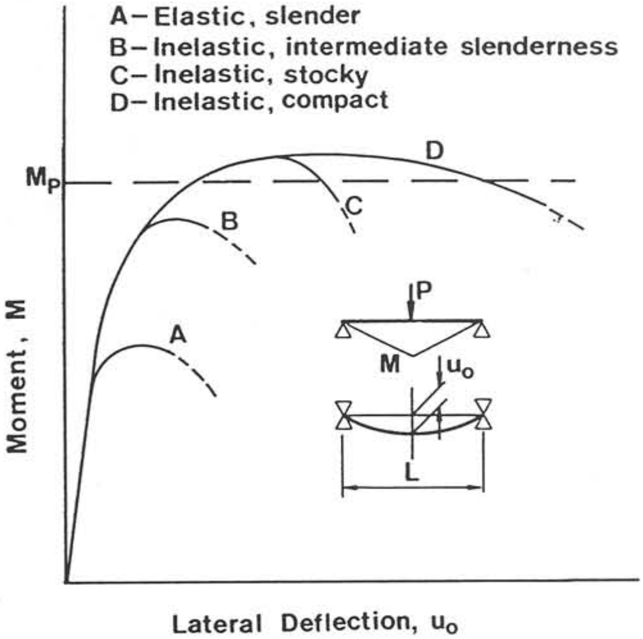


Fig. 2 — Load-Deformation Behaviour of Typical I-Beam.

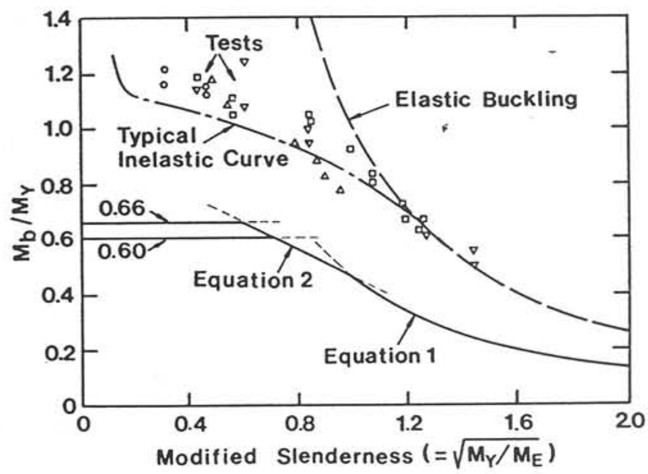


Fig. 3 — Design Rules in AS 1250-1981.

equilibrium. These are referred to as bifurcation-of-equilibrium analyses. At the bifurcation load the beam is theoretically capable of maintaining a buckling mode shape characterized by lateral displacement and twist. Experiments have shown that real beams with geometrical imperfections exhibit buckling type displacements from the onset of loading. The experimental buckling load is essentially the maximum attainable load at which buckling type displacements become large. Fig. 3 shows the capacity versus slenderness behaviour of a beam which indicates that geometrical and material imperfections reduce capacities from the ideal, particularly as slenderness reduces.

1.3 PRESENT DESIGN RULES

Many codes of practice and design proposals (1, 2, 3) relate the strength of a beam to its elastic buckling capacity (1, 3). The current Australian Steel Structures Code AS 1250-1981 (1) uses a set of semi-empirical equations to relate the allowable moment of a beam, M_b, to its first yield moment, M_Y and its elastic buckling moment, M_E. For slender beams, when M_E is less than M_Y,

$$M_b/M_E = 0.55 - 0.10\, M_E/M_Y \qquad (1)$$

and for intermediate length beams, when $M_E \geqslant M_Y$

$$M_b/M_Y = 0.95 - 0.50\sqrt{M_Y/M_E} \qquad (2)$$

- in which $M_b/M_Y \not> 0.60$ or 0.66 depending on the width-to-thickness ratio of the compression flange outstand. Equations (1) and (2) are compared with typical inelastic buckling curves and with experimental results in Fig. 3. It is seen that these equations provide a safety factor varying from 1.67 for short beams to 1.9 for long beams. To achieve factors of safety in this range, designers need accurate values of M_E.

In determining the elastic buckling moment, AS 1250-1981 gives designers the option of either obtaining M_E from an elastic buckling analysis, which is not always possible in design offices, or by using approximate formulae or tables in the Code. These formulae or tables are based on a number of simplifying assumptions and were derived for a simply supported beam under uniform moment which is considered the worst loading condition. Although some guidelines are given in the Code for estimating effective length factors for beams under various loading and restraint conditions, these are very crude.

The purpose of this paper is to present designers with accurate elastic buckling solutions for beams subjected to common loading and restraint conditions, so that a more economical design may be achieved.

1.4 SCOPE

The following Sections summarize many available elastic buckling solutions obtained from bifurcation analyses. The solutions are applicable to uniform beams of doubly symmetric cross-section such as I-beams. Information on the buckling of monosymmetric beams and non-uniform beams can be obtained from References 4 to 8.

Section 2 considers simply supported beams and cantilevers under moment gradient loading, point loading and uniformly distributed loading. Solutions are also presented for single span beams with a variety of end restraint conditions. The influence of the level of application of transverse loading is also examined.

Section 3 examines elastic beams and cantilevers with internal bracing. The influence of brace type and location is investigated along with that of the level of application of load above and below the shear centre. Moment gradient loading, point loading and uniformly distributed loading are considered. The Section provides data on optimum internal brace location.

Section 4 presents a general method for determining the elastic buckling loads of laterally continuous structures. These range from beams with multiple supports and braces to grid structures. An analysis procedure and worked examples are presented.

2. Buckling of End Restrained Beams

2.1 INTRODUCTION

This Section gives lateral buckling capacities for single segment beams under a variety of major axis loading and end restraint conditions. The data applies to elastic beams which are initially straight and are of doubly symmetric cross-section. Additional information can be found from the references cited.

2.2 SIMPLY SUPPORTED BEAMS WITH END MOMENTS

The basic loading configuration is shown in Fig. 4. Also shown are characteristic buckling displacements: lateral displacement, u, and twist, ϕ. The simple supports prevent twist and lateral movement at ends A and B but warping and minor axis rotation are free to occur. The moment ratio, β, lies in the range $-1 \leqslant \beta \leqslant +1$ with $\beta = -1$ for uniform bending. The elastic buckling solutions limit the value of the larger end moment, M, which occurs at end A.

2.2.1 Uniform Moment Loading ($\beta = -1$)

The elastic buckling moment for a beam in uniform bending is

$$M_O = \pi/L\sqrt{EI_yGJ}\left[\sqrt{1+K^2}\right] \qquad (3)$$

in which EI_y = minor axis bending rigidity; GJ = St Venant torsional rigidity; EI_ω = warping rigidity; L = beam length and K = beam parameter,

$$K = \sqrt{\pi^2 EI_\omega / GJL^2} \qquad (4)$$

The beam parameter, K, is a measure of the beam's ability to resist non-uniform torsion via internal warping restraint along its length. For practical purposes it ranges from K = 0 (narrow rectangular beam) to K = 3.0 (stocky I-beam). Fig. 5 indicates the variation of K with cross-sectional properties and length for I-beams.

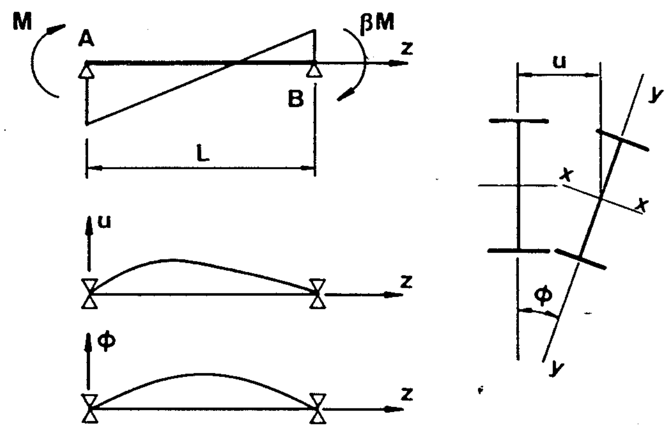


Fig. 4 — Simply Supported Beam with End Moments.

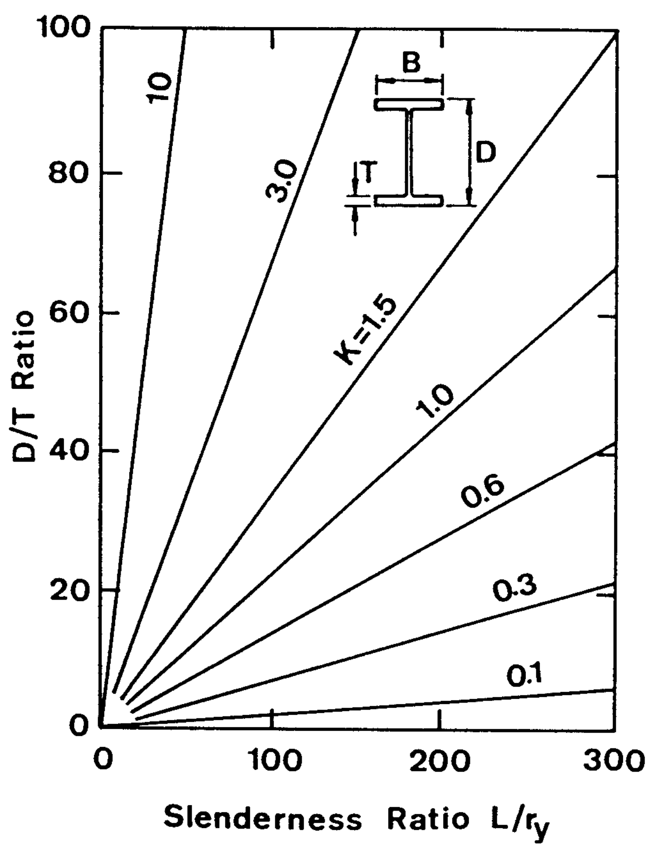


Fig. 5 — Variation of Beam Parameter, K.

Loading	Modification Factor, m*	Notes
M, βM	$1.75+1.05\beta+0.3\beta^2$	≤ 2.56 $-1 \leq \beta \leq +1$
aL, P	$1.9-2.2a(1-a)$	$0 \leq a \leq .5$
aL, P, P, aL	$1-0.4a(1-5.5a)$	$0 \leq a \leq .5$
$q\frac{PL}{8}$, P, $q\frac{PL}{8}$	$1.35+0.36q$	$0 \leq q \leq 1$
$q\frac{wL^2}{12}$, w, $q\frac{wL^2}{12}$	$1.13+0.12q$ $4.8q-2.38$	$0 \leq q \leq .75$ $.75 \leq q \leq 1$
M	0.25	K=0
P	1.28	K=0
w	2.05	K=0

*Shear Centre Loading

Fig. 6 — Moment Modification Factors for Beams and Cantilevers.

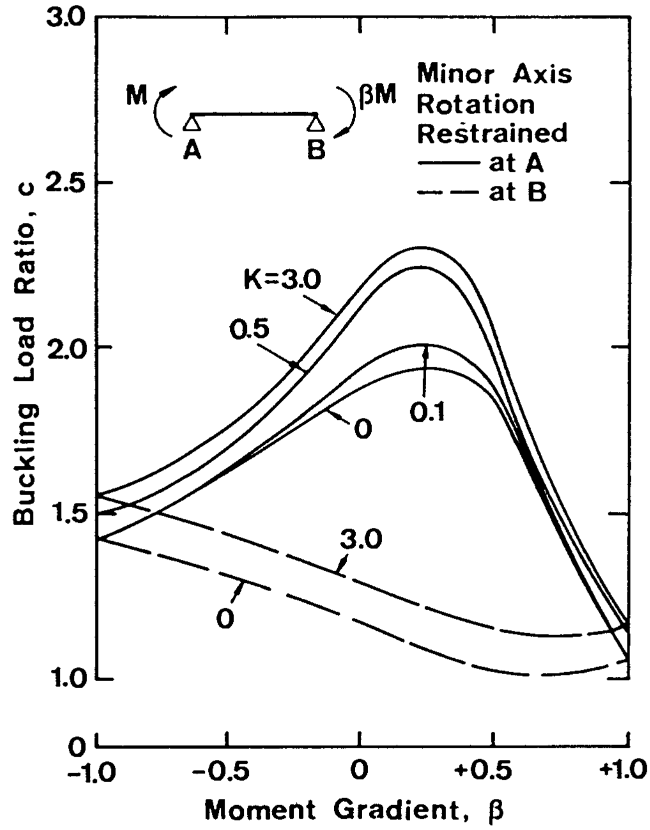


Fig. 7 — Buckling Load Ratios for Moment Gradient Beams with One End Restrained Against Minor Axis Rotation.

2.2.2 Moment Gradient Loading ($-1 \leqslant \beta \leqslant +1$)

No closed form expressions are available for $\beta > -1$. Approximate capacities can be expressed in terms of Equation (3) and a moment modification factor, m, found by numerical solution of the governing differential equations (e.g. Reference 3).

$$M = M_E = mM_o \tag{5}$$

in which

$$m = 1.75 + 1.05\beta + 0.3\beta^2 \leqslant 2.56 \tag{6}$$

and M is the larger end moment. Fig. 6 lists moment modification factors for a range of beams and cantilevers. A more detailed presentation of m is offered in the following Sections.

2.3 BEAMS WITH END MOMENTS AND ADDITIONAL END RESTRAINTS

This Section considers moment gradient beams with end restraints in addition to simple supports. End warping restraints and minor axis rotational restraints are examined separately and together at one or both ends. The restraint stiffness is assumed to be large enough to fully restrain the action in question.

Moment capacities can be found from

$$M = c.M_E \tag{7}$$

in which c = buckling load ratio from Figs. 7 to 11.

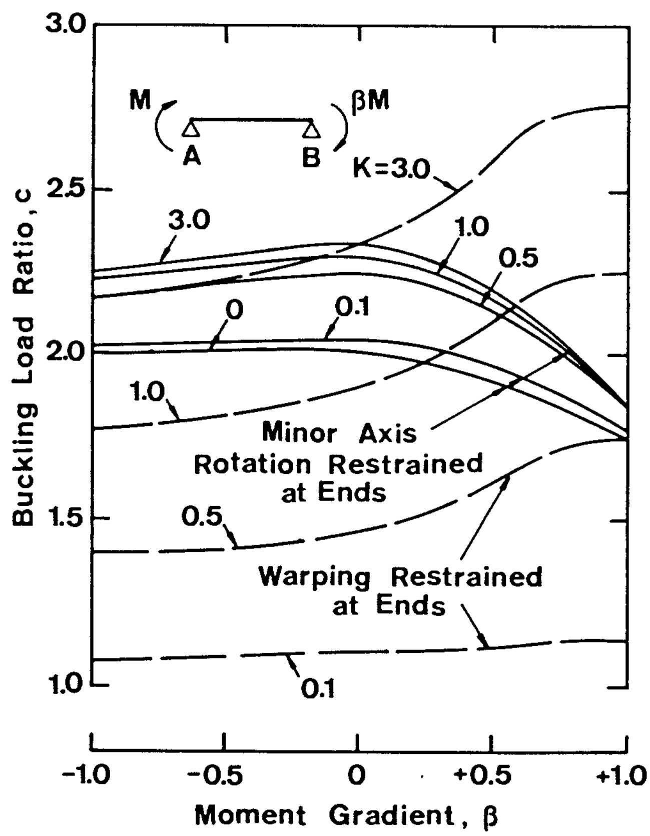


Fig. 8 — Buckling Load Ratios for Moment Gradient Beams with Both Ends Restrained against Minor Axis Rotation or Warping.

2.3.1 Minor Axis Rotational Restraint

Fig. 7 indicates buckling load ratios due to a rotational restraint at either end. The restraint is most effective

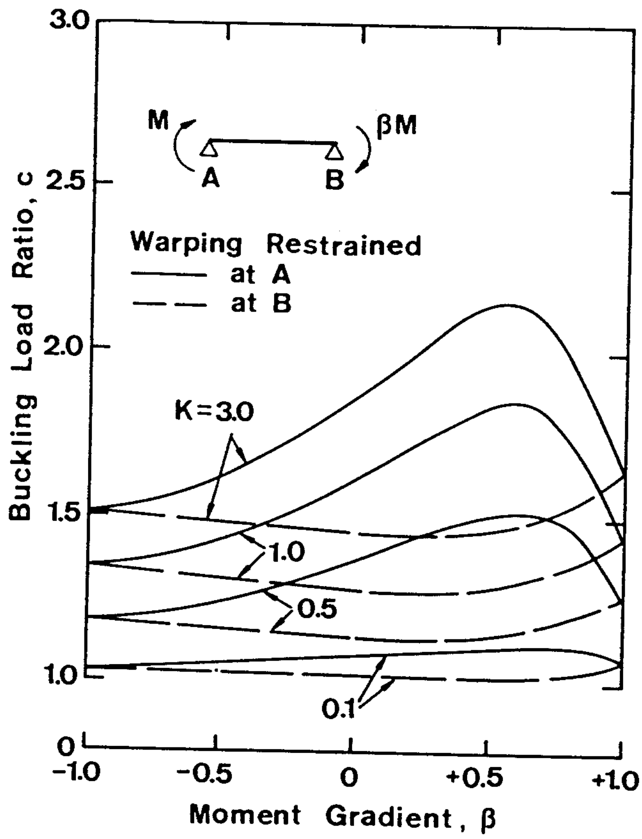


Fig. 9 — Buckling Load Ratios for Moment Gradient Beams with One End Restrained Against Warping.

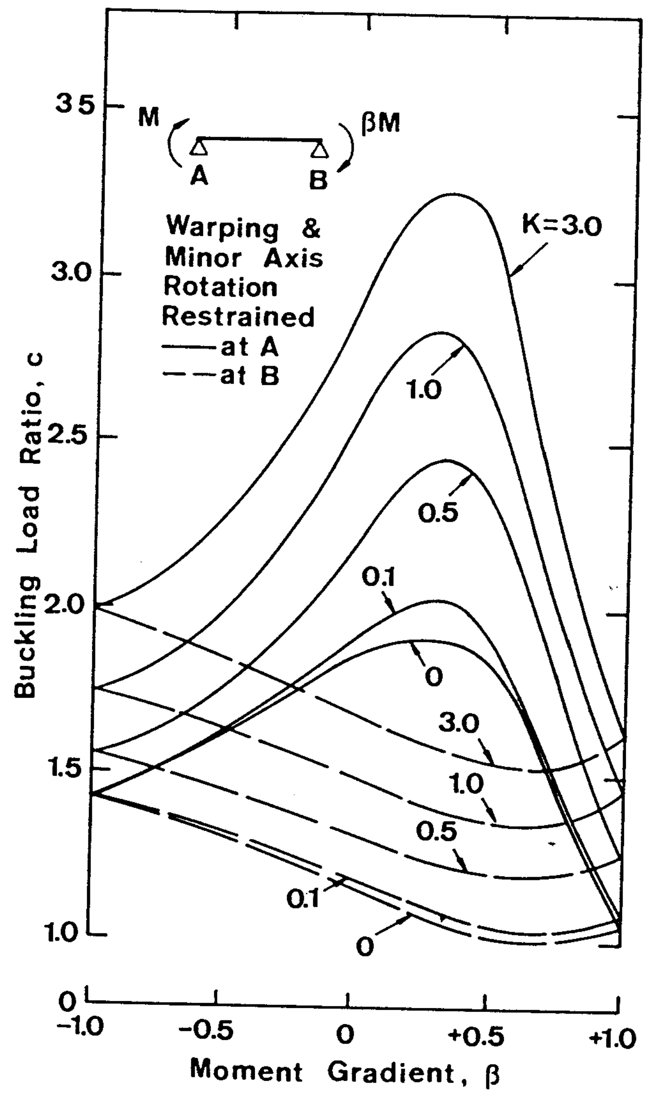


Fig. 10 — Buckling Load Ratios for Moment Gradient Beams with One End Restrained Against Warping and Minor Axis Rotation.

when placed at end A (moment = M) and when $-0.25 \leqslant \beta \leqslant 0.5$. A restraint at end B (moment = βM) becomes less effective as β increases. When $\beta \approx 0.75$ the buckling mode shape, u, has a zero derivative, du/dz, at end B and a restraint at B has no influence. Buckling load ratios in excess of unity in this region of β reflect inaccuracies in Equation (6) rather than any beneficial effect of the restraint.

Fig. 8 shows that minor axis rotational restraints at both ends increase the buckling load by a factor ≈ 2 over the full β and K ranges. Figs. 7 and 8 indicate a moderate variation of c with K.

2.3.2 Warping Restraint

Buckling load ratios for beams with a warping restraint at either A or B are given in Fig. 9. The restraint is best placed at end A and is most effective for higher moment gradients. A warping restraint at end B produces a lesser and more uniform buckling load ratio. The modal warping displacement distribution, $d\phi/dz$, and its derivative, $d^2\phi/dz^2$, are less sensitive to changes in β than are u and du/dz. Of particular influence is the beam parameter K. Small K implies a beam which resists torsion mainly by developing St Venant shearing stresses. The restraint of end warping has thus only a marginal effect on torsional stiffness and hence on buckling capacity.

Warping restraints at both ends result in the buckling load ratios in Fig. 8. Again K is of importance. References 9 and 10 provide warping restraint details and design methods.

2.3.3 Combined Warping and Rotational Restraint

Figs. 10 and 11 show the effect of combined restraints at either or both ends. For a single combined restraint, the restraint location, moment gradient and beam parameter are of influence. Combined restraints at both ends produce buckling load ratios which depend mainly on K.

2.4 SIMPLY SUPPORTED BEAMS WITH TRANSVERSE LOADING

2.4.1 Central Point Load

Moment modification factors, m, for loading at three levels; top flange (TF), shear centre (SC) and bottom flange (BF), are given in Fig. 12. These factors are to be used with Equation (5) where M_O is found from Equation (3). The buckling moment thus obtained limits the maximum moment which, in this instance, occurs at mid-span. The loads are free to sway laterally hence load application level is of importance as is the beam parameter. In comparison to a load at the shear centre, top flange loading is more severe as it accentuates twisting whereas a bottom flange load opposes it. The range of m is therefore large.

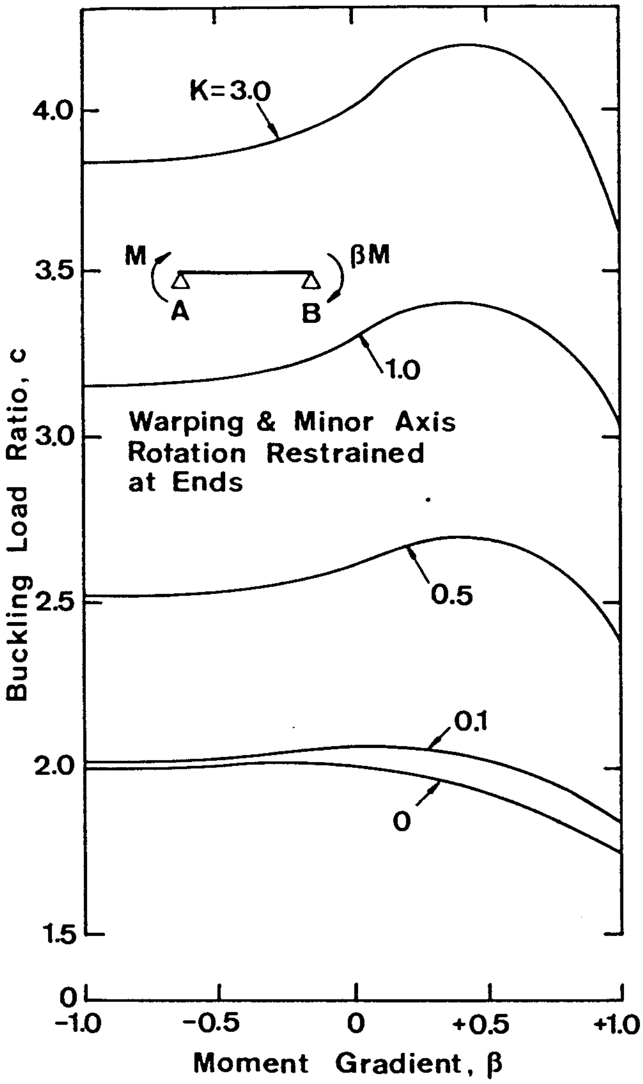


Fig. 11 — Buckling Load Ratios for Moment Gradient Beams with Both Ends Restrained Against Warping and Minor Axis Rotation.

2.4.2 Uniformly Distributed Load

Moment modification factors for top flange, shear centre and bottom flange loading are given in Fig. 13. Equation (5) provides a limit to the mid-span moment. Again, top flange loading is seen to be most severe and a substantial range of m is evident. This range increases with increasing K but is reasonably constant for $K > 2.0$. References 11 and 12 provide additional data.

2.5 BEAMS WITH TRANSVERSE LOADING AND ADDITIONAL END RESTRAINTS

Beams with transverse loading and end restraints in addition to simple supports are considered here. Warping restraints and minor axis rotational restraints are examined separately or together at both ends. Reference 11 provides additional data.

2.5.1 Minor Axis Rotational Restraint

Figs. 14 and 15 give buckling load ratios for beams with both ends restrained against minor axis rotation and carrying the transverse loadings considered in Section 2.4. The ratios are sensitive to load level and beam parameter. They are to be used with Equation 7.

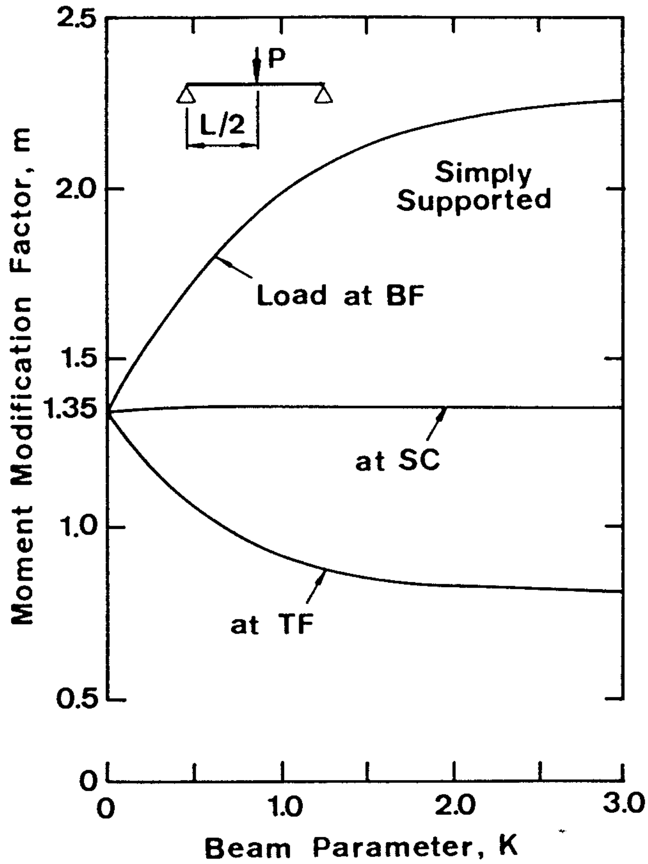


Fig. 12 — Moment Modification Factors for Simply Supported Beams with Central Point Load.

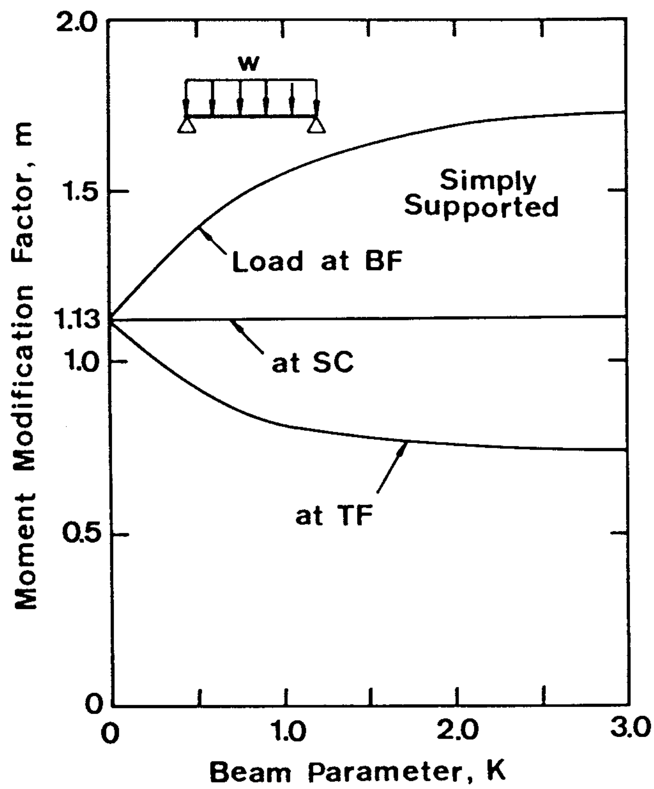


Fig.13 — Moment Modification Factors for Simply Supported Beams with Uniformly Distributed Load.

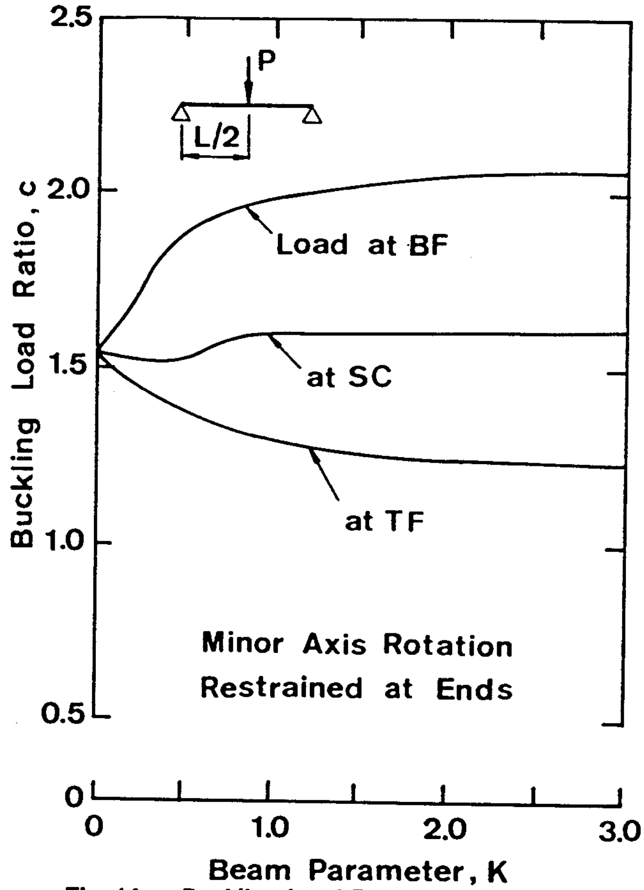


Fig. 14 — Buckling Load Ratios for Minor Axis Rotation End Restrained Beams with Central Point Load.

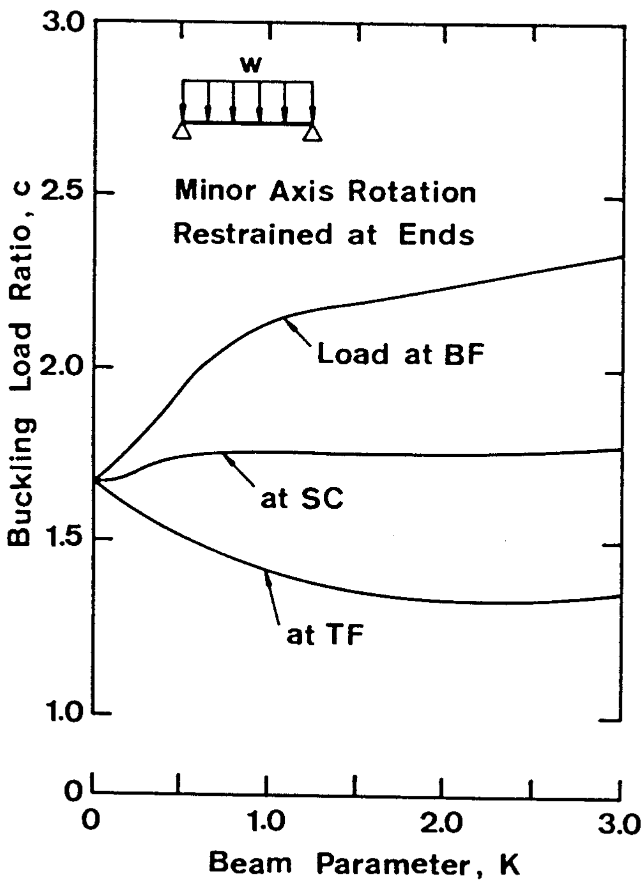


Fig. 15 — Buckling Load Ratios for Minor Axis Rotation End Restrained Beams with Uniformly Distributed Load.

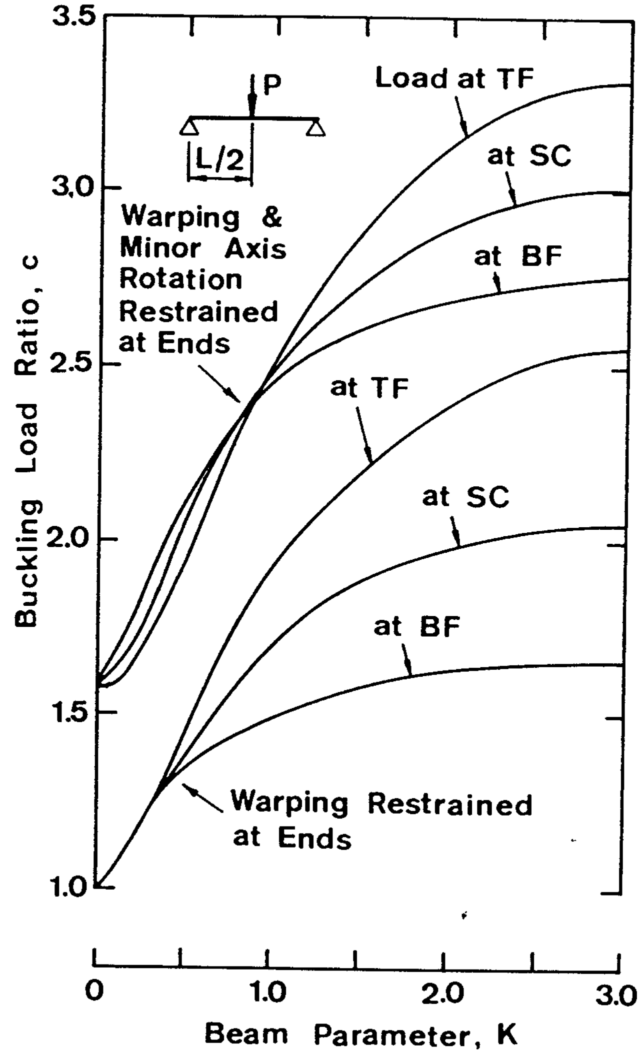


Fig. 16 — Buckling Load Ratios for Warping and Minor Axis Rotation End Restrained Beams with Central Point Load.

2.5.2 Warping Restraint

End warping restraints in addition to simple supports produce the buckling load ratios in Figs. 16 and 17. As K increases, the additional torsional stiffness afforded to the beam by the restraint of end warping has a most pronounced influence when loading is at top flange level. The buckling load ratios are highest for loading at this level. This occurs for both transverse load types. The combined factor, cm, is highest for bottom flange loading.

2.5.3 Combined Warping and Minor Axis Rotational Restraint

An effect similar to that described in Section 2.5.2 is seen in Figs. 16 and 18 for combined warping and rotational restraints at both ends. The buckling load ratio is sensitive to K for $K < 1.0$ and approaches a more uniform value as K tends to 3.0.

2.6 PROPPED CANTILEVERS WITH TRANSVERSE LOADING

Moment modification factors are presented in Fig. 19 for propped cantilevers carrying a central point load or a uniformly distributed load. Warping and minor axis

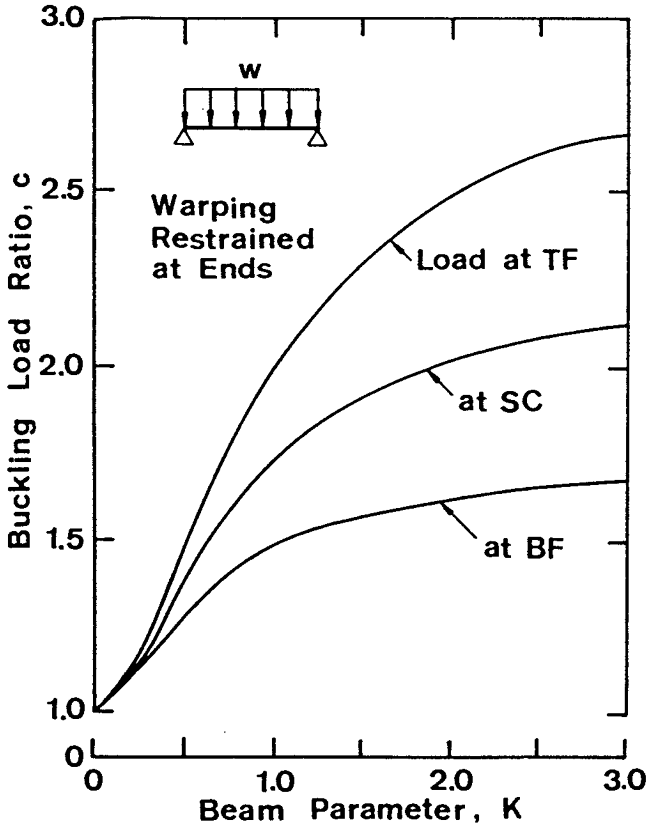


Fig. 17 — Buckling Load Ratios for Warping End Restrained Beams with Uniformly Distributed Load.

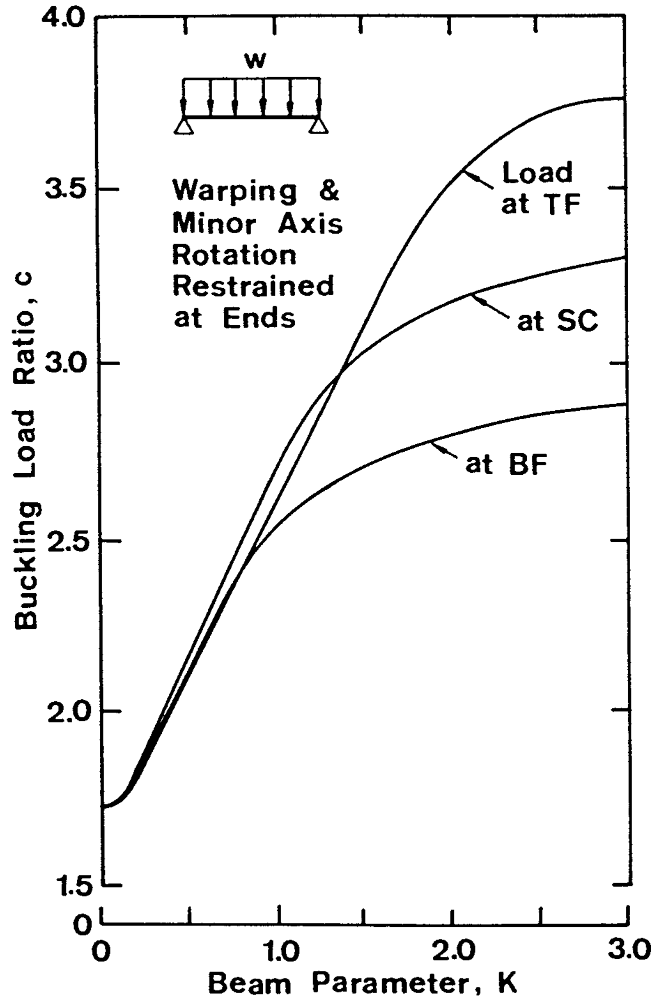


Fig. 18 — Buckling Load Ratios for Warping and Minor Axis Rotation End Restrained Beams with Uniformly Distributed Load.

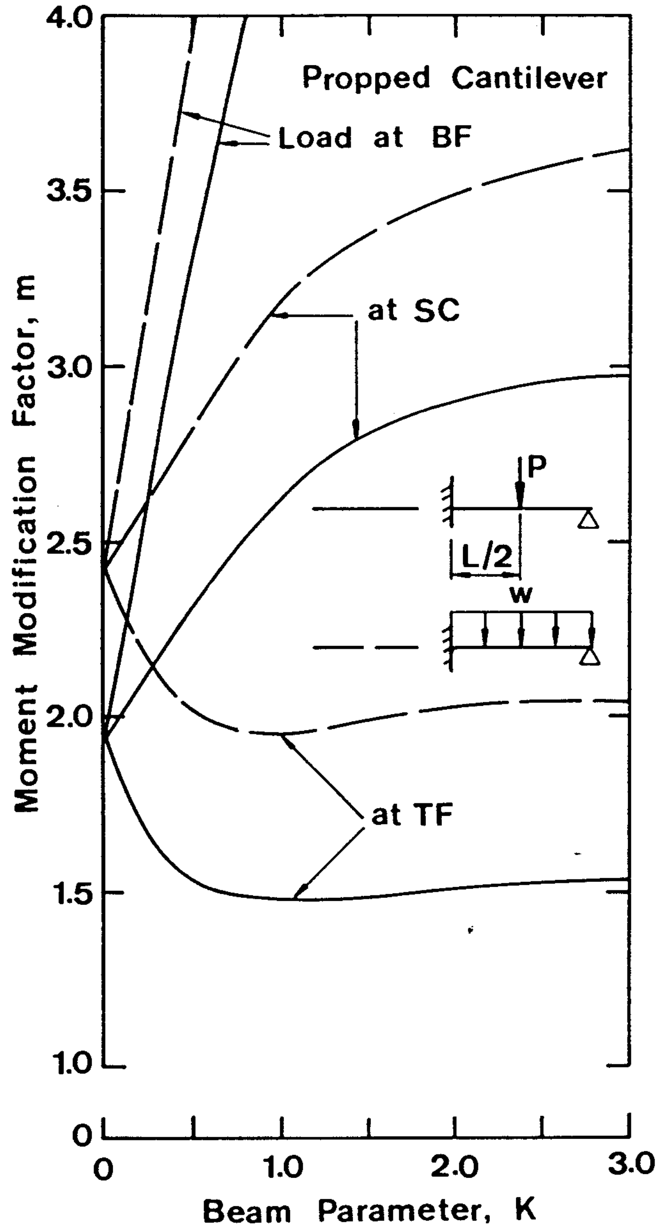


Fig. 19 — Moment Modification Factors for Propped Cantilevers.

rotation are prevented at the fixed end along with major axis rotation. Conditions at the prop are assumed to prevent twist and lateral movement. Equation (5) limits the moment at the fixed end for both loadings. Three loading levels are considered and top flange loading is seen to be the most severe. The total distributed load is larger than the point load in all corresponding instances. Further information can be found in Reference 12.

2.7 FIXED-ENDED BEAMS WITH TRANSVERSE LOADING

A central point load and uniformly distributed loading are considered at top and bottom flange levels and at the shear centre. All movement is prevented at the fixed ends. Moment modification factors in Fig. 20 along with Equation (5) limit the moments at these ends. As expected top flange loading is most severe and point loading is less favourable for stability than is distributed loading. Modification factors for shear centre and bottom flange loading increase rapidly with K. Reference

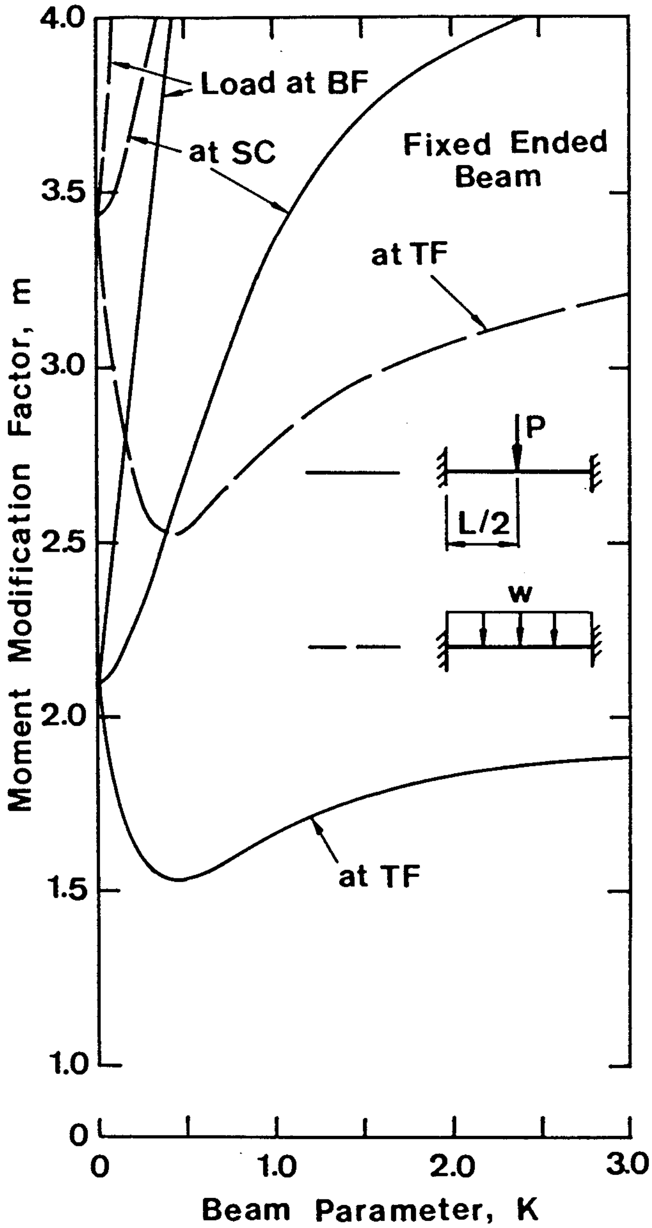


Fig. 20 — Moment Modification Factors for Fixed Ended Beams.

13 provides capacities for beams under variable major axis rotational end restraints.

2.8 CANTILEVERS WITH TRANSVERSE LOADING

Fig. 21 provides moment modification factors for cantilevers with tip loading or uniformly distributed loading at three levels of application. The cantilevers are free to move at all locations other than at the fixed end. Buckling capacities from Equation (5) limit the moment at this end. The modification factors vary with K and reduce dramatically for top flange loading. Note that the modification factors for transversely loaded cantilevers in Fig. 6 apply only when K = 0. Additional data are presented In References 14 and 15.

2.9 CONCLUSION

This Section has dealt with the stability of simply supported beams, cantilevers and beams with a variety of additional end restraints and under a number of different loadings. Several parameters are of influence,

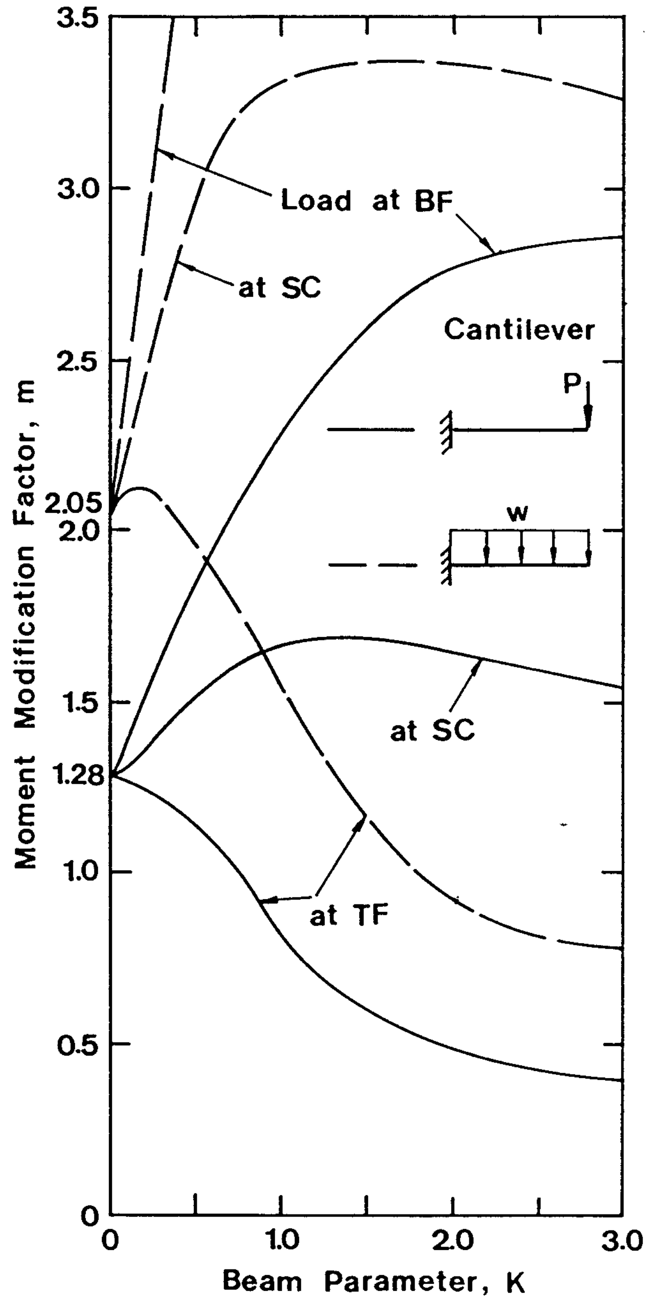


Fig. 21 — Moment Modification Factors for Cantilevers.

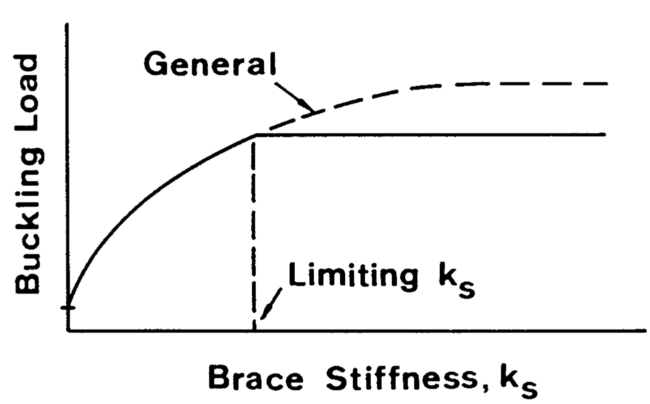


Fig. 22 — Typical Relationship Between Buckling Load and Brace Stiffness.

notable the beam parameter, moment gradient and level of application of transverse loading. The benefits of judiciously chosen end restraints are obvious from the many figures. Section 3 discusses the stability of beams and cantilevers with braces along their length. Buckling load ratios from that Section are to be used with modification factors from Section 2 when assessing buckling loads.

3. Bracing of Beams

3.1 TYPES OF BRACING

Bracing may be provided by either a continuous medium such as a diaphragm or by one or more discrete braces along the beam length. Usually it is not economical to provide anything less than 'rigid' bracing for a member. To achieve a condition of rigid bracing, the brace must possess certain minimum stiffness. A typical relationship between the ratio of the elastic buckling load of a braced beam and the stiffness of the brace, k_S, is shown in Fig. 22. For members under uniform moment with a central brace, there is a limiting value of the brace stiffness. In general the curve rises slowly with increasing k_S, and there is no distinct discontinuity in the curve. However, the limiting stiffness can be defined as one which is close to that corresponding to rigid bracing. In practice, it is not difficult to achieve adequate bracing stiffness and code rules for assessing this requirement are available (References 1, 2 and 16). In this Section, adequate stiffness is assumed for the types of braces considered.

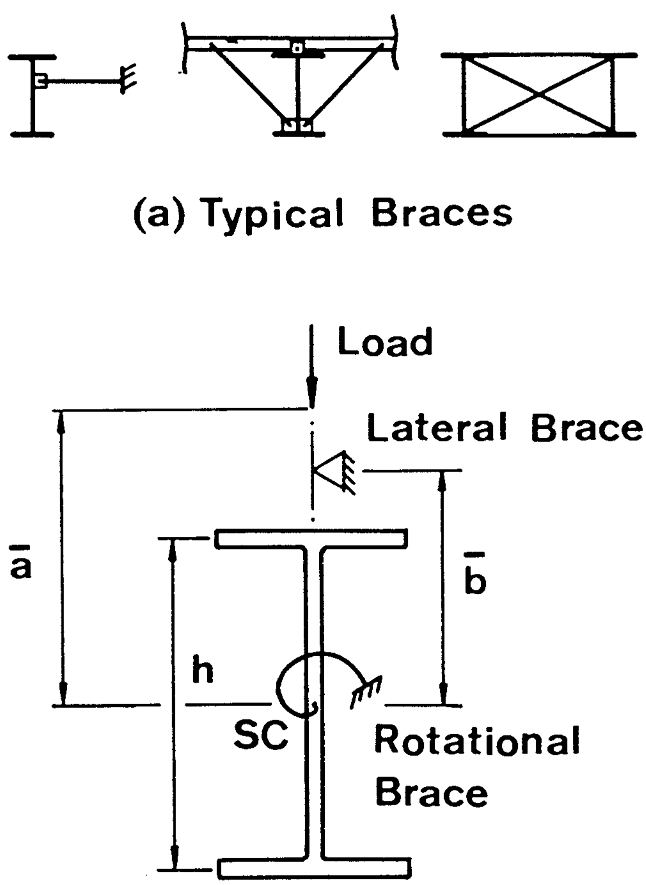


Fig. 23 — Cross-section at Brace.

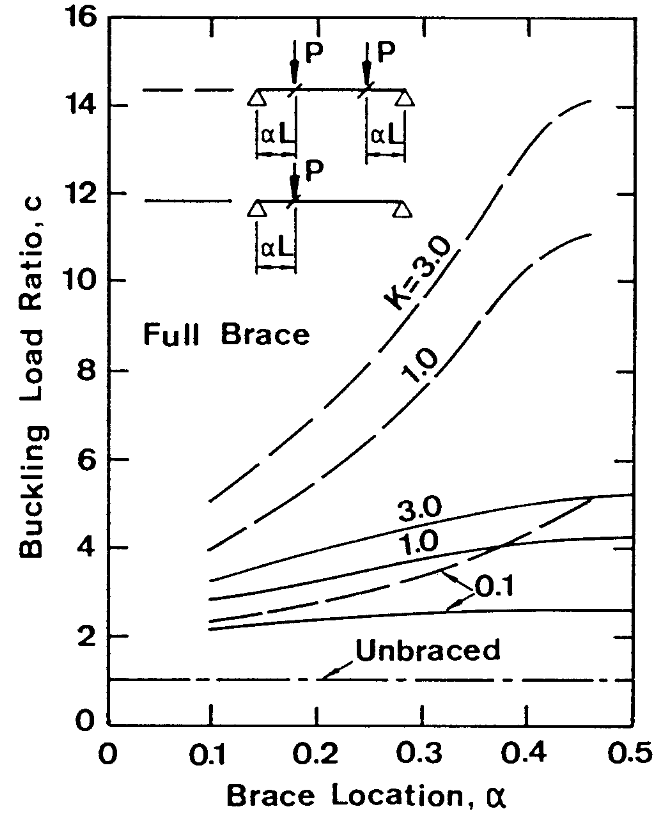


Fig. 24 — Buckling Load Ratios for Simply Supported Beams with One or Two Point Loads and Full Braces.

While beams may be braced in many different ways, most arrangements can be represented by an idealized system comprising an elastic lateral brace acting at a distance $\bar{b}$ above the shear centre of the beam cross-section and an elastic rotational brace (see Fig. 23). Full bracing effectively prevents lateral deflection and twist at the cross-section, while partial bracing allows some limited twisting or lateral deflection to occur. The term lateral bracing is used to describe bracing which prevents lateral deflection at the point where the brace is placed, while rotational bracing prevents twisting only.

The increase in the buckling capacity in a beam due to the particular brace is defined by the buckling load ratio, c :

$$c = \frac{\text{buckling capacity of the braced beam}}{\text{buckling capacity of a similar unbraced beam}} \quad (8)$$

Unbraced beam capacities can be found in Section 2. Results presented in the following Sections have been obtained using the method of finite integrals (Reference 17) to solve the governing differential equations for bending and torsion (Reference 18). Braces were included by imposing appropriate boundary conditions involving lateral deflection, u, and cross-sectional rotation, ϕ.

3.2 SIMPLY SUPPORTED BEAMS

3.2.1 Braced Beams with Point Loads

The buckling of simply supported beams with one or two point loads acting at the shear centre is examined in Fig. 24. It is common in practice for load points to be braced and hence the height of application of the load above the

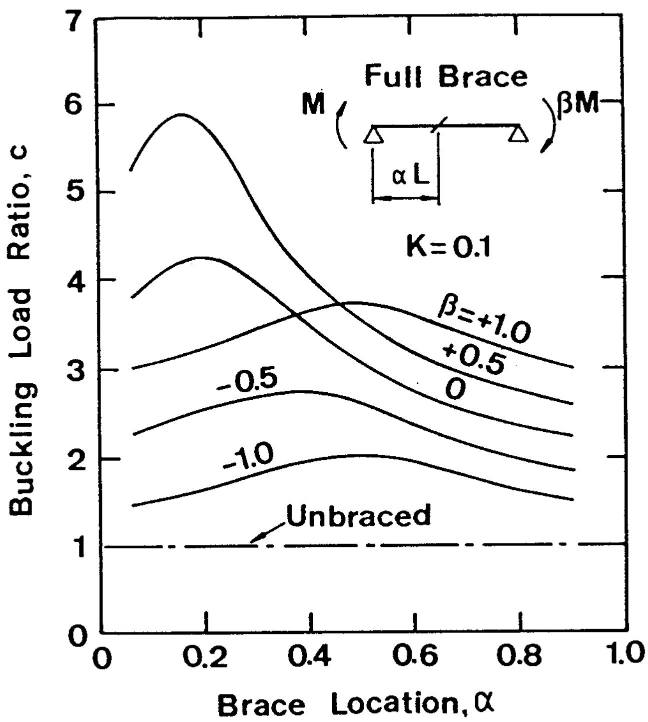


Fig. 25 — Buckling Load Ratios for Simply Supported Beams with End Moments and a Full Brace, K = 0.1.

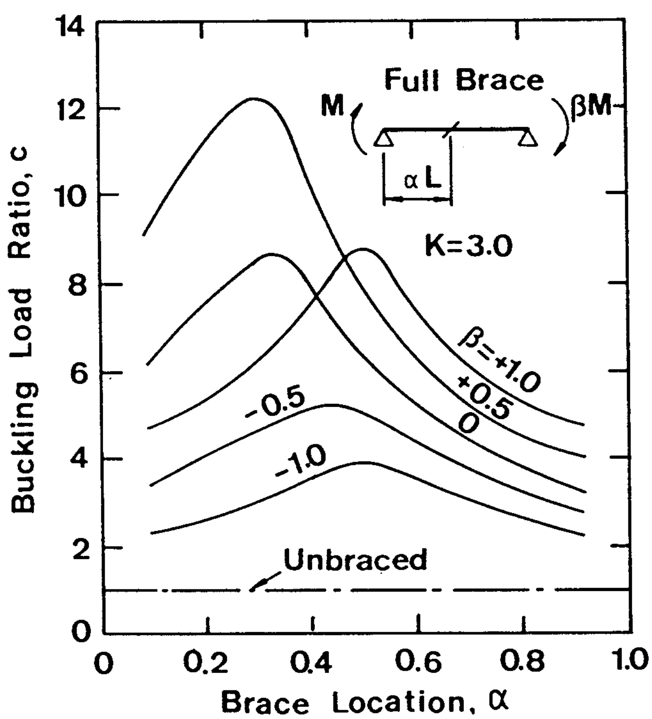


Fig. 26 — Buckling Load Ratios for Simply Supported Beams with End Moments and a Full Brace, K = 3.0.

shear centre, a, does not affect beam buckling capacity. Buckling load ratios, c, are shown in Fig. 24 for three values of beam parameter, K, and for one or two point loads at location, α.

3.2.2 Braced Beams with End Moments ($-1 \leqslant \beta \leqslant +1$)

Simply supported beams with end moments are considered in this Section. The effect of a full brace at

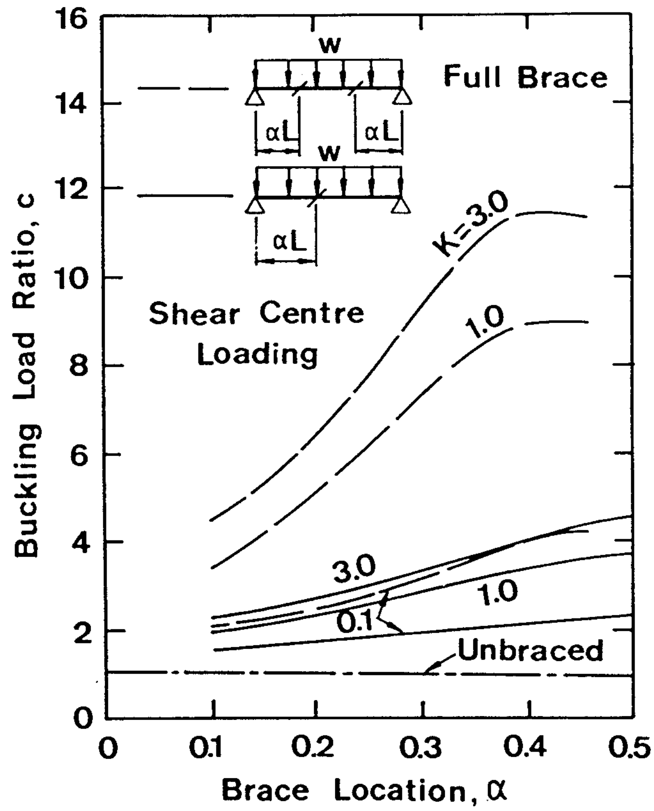


Fig. 27 — Buckling Load Ratios for Simply Supported Beams with Uniformly Distributed Load and One or Two Full Braces.

various locations along a beam is shown respectively, in Figs. 25 and 26 for K = 0.1 and 3.0. The curves indicate an optimum brace location for each in-plane moment distribution.

3.2.3 Braced Beams with Uniformly Distributed Load

Simply supported beams with uniformly distributed loads and with one or two intermediate braces are considered in Figs. 27 and 28. This type of loading and bracing is common in roof structures where the distributed load may arise from wind or live loading. The load may act at the top flange, shear centre, bottom flange or at any other level. It is usual to provide fly-bracing to the bottom flange of a roof beam under uplift in order to increase its lateral buckling capacity.

Solutions for simply supported beams with distributed loads acting at the shear centre are shown in Fig. 27. These results are similar to those obtained for beams with point loads (see Fig. 24) and demonstrate the considerable influence of the beam parameter K on the value of c. The effect of the level of load application is shown in Fig. 28 for K = 1.0. The increase in the buckling load ratio is more pronounced for a beam with top flange loading. Unbraced loads applied above the shear centre contribute to the destabilizing influence of in-plane moments, hence appropriately placed braces are particularly effective.

Figs. 27 and 28 show an increase in the buckling load ratio as the brace is moved towards the beam centre. A single brace is most effective when placed at mid-span.

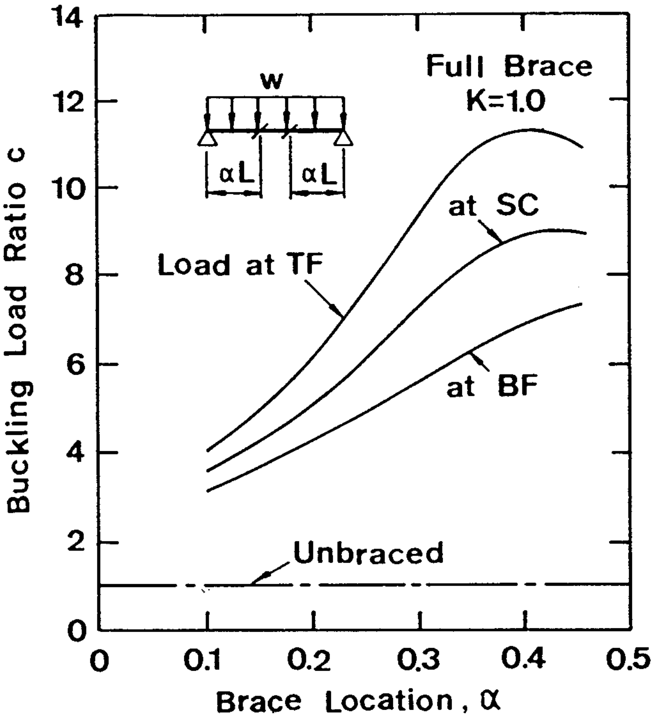


Fig. 28 — Influence of Load Height on Simply Supported Beams with Uniformly Distributed Load and Two Full Braces.

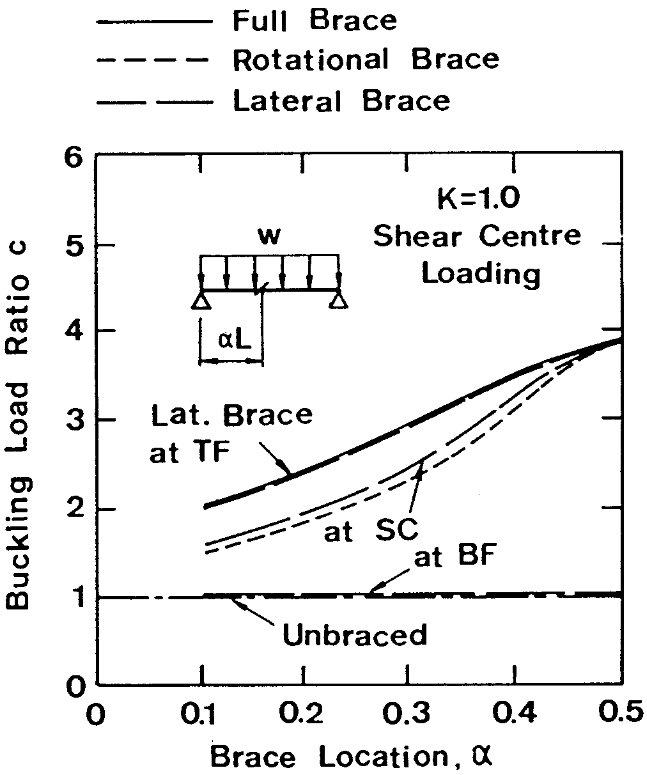


Fig. 29 — Comparison of Brace Types for Simply Supported Beams with Uniformly Distributed Load, K = 1.0.

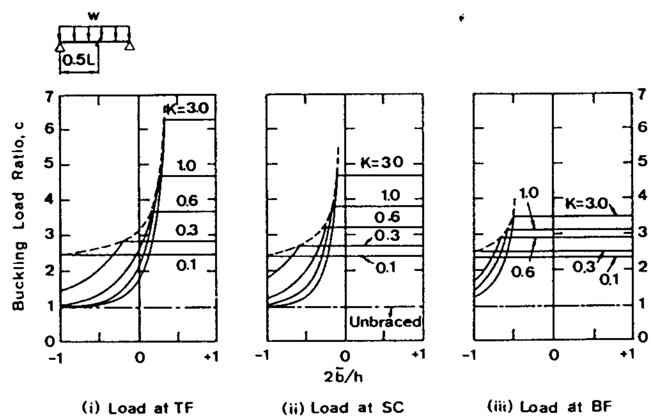


Fig. 30 — Influence of Lateral Brace and Load Height for Simply Supported Beams.

Two braces in close proximity combine to restrain minor axis rotation and this is evident as the two approach mid-span. Contrary to the common practice of bracing the third points of a beam with uniformly distributed load, it is better to locate braces at the 2/5 points ($\alpha \approx 0.4$). This leads to an increase in the elastic buckling loads of 10 to 15 percent.

3.2.4 Effect of Lateral or Rotational Bracing

Simply supported beams with lateral bracing or rotational bracing alone are considered. Results for beams with uniformly distributed load are compared in Fig. 29. Beams with a lateral brace attached to the top flange, the shear centre, or the bottom flange are compared to beams with a rotational brace at the shear centre, and to beams with a full brace. Lateral bracing is most effective when acting at the top or compression flange level. When attached to the bottom or tension flange, the brace in this instance is completely ineffective. A rotational brace is best located near mid-span.

In Fig. 30, the effect of the position of the lateral brace above and below the shear centre is considered with load acting at the top flange, the shear centre or the bottom flange. Solutions have been obtained for a number of beam parameters and with the brace at mid-span (α = 0.5). Results indicate that the influence of the brace depends on the brace level ($2\bar{b}/h$), the load height ($2\bar{a}/h$) and also the beam parameter K. Generally it is best to attach the lateral brace to the compression side. For small values of K, the position of attachment is unimportant, and the brace is quite effective even if placed at the tension flange level. However, for larger values of K, the region suitable for brace attachment diminishes. This reduction becomes more severe if the load acts on the compression flange.

3.3 CANTILEVERS

3.3.1 Braced Cantilevers

The influence of the position of a full brace is examined. The buckling load ratio, c, for values of the beam parameters K = 0.1 to 3.0 are shown in Figs. 31 and 32 for cantilevers with a tip load and uniformly distributed load respectively. The loads are applied at top flange, shear centre or bottom flange. It can be seen that the increases in the buckling load are greatest for large values of the beam parameter and more so for top flange loading. The maximum value of c that may be achieved ranges from 3 for small values of K to 14 for large values of K.

The results show that for small values of K the optimum brace location is near mid-span for a tip load, and near

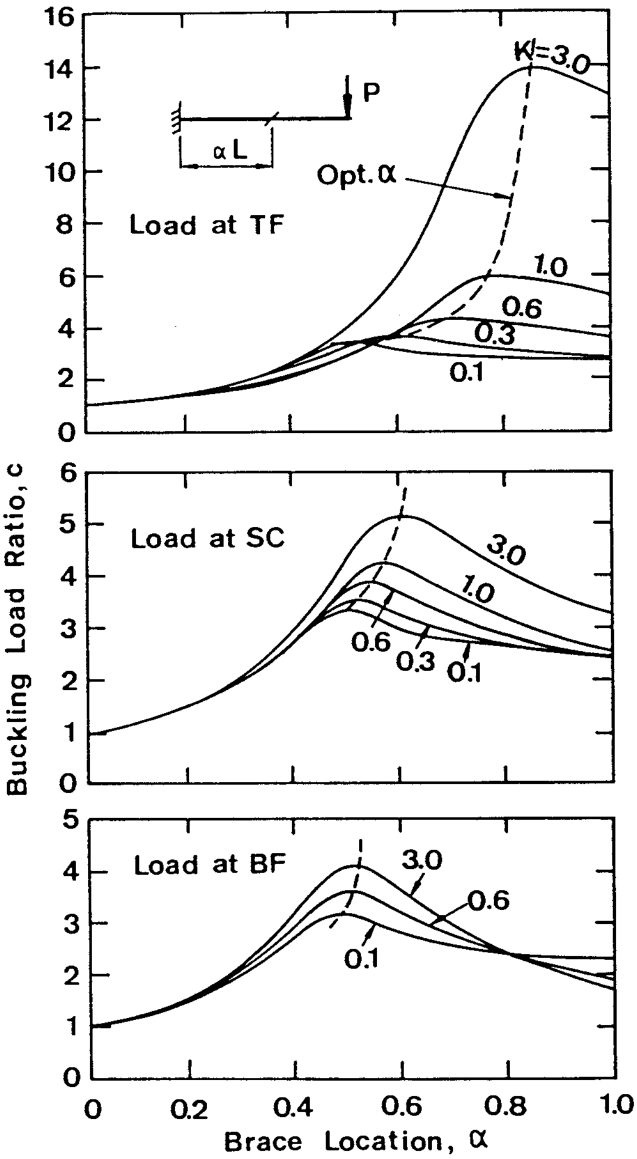


Fig. 31 — Buckling Load Ratios for Cantilevers with a Tip Load and a Full Brace.

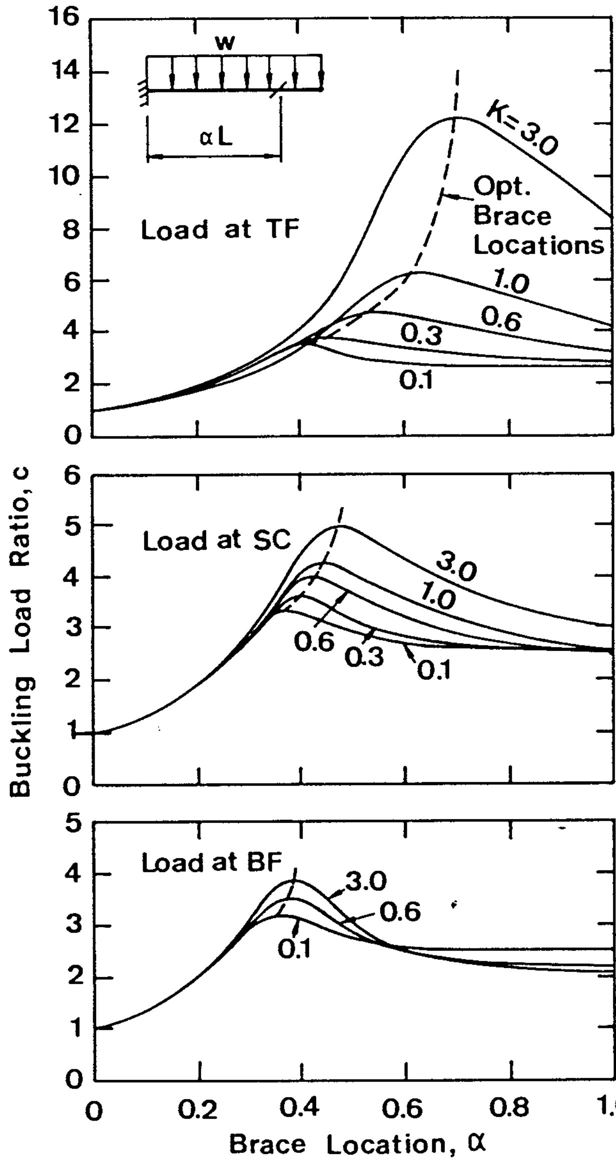


Fig. 32 — Buckling Load Ratios for Cantilevers with Uniformly Distributed Load and a Full Brace.

0.4 of the length from the fixed end for a uniformly distributed load. For higher values of K, the optimum brace locations approach the cantilever tip as the height of load application moves toward the top flange. For a tip load, the optimum location varies between $\alpha = 0.5$ and 0.8 and for a uniformly distributed load, between $\alpha = 0.4$ and 0.7.

3.3.2 Effects of Lateral and/or Rotational Bracing

The effectiveness of lateral bracing at various levels of attachment is compared with that of rotational bracing and of full bracing for top flange, shear centre and bottom flange loading, for values of K = 0.6 and 3.0. The results are shown respectively, in Figs. 33 and 34 for tip load and in Figs. 35 and 36 for uniformly distributed load.

The various braces have different influences on the buckling load depending on the level of load height ($2\bar{a}/h$). In all cases full bracing is by far the best. If full bracing cannot be achieved, rotational bracing is the next best as can be seen from Figs. 33 to 36. Optimum brace locations are clearly evident.

For lateral bracing only, the buckling load ratios increase slowly as the brace moves towards the tip, irrespective of the level of the brace. Varying the value of K has only little effect on the maximum value of c for top flange and bottom flange loadings. However, for shear centre loading and with a top flange brace, the effect of increasing K shows a marked improvement in the value of c.

Lateral braces should be placed as close as possible to the cantilever tip. The effectiveness increases as the level of application of load moves towards the bottom flange. In all cases if lateral bracing alone is used, it should be placed near the top flange and as close as possible to the cantilever tip. Braces placed less than 0.4L from the fixed end are practically useless.

3.4 CONCLUSION

The beam parameter K, has a significant influence on the elastic buckling strength of simply supported beams and cantilevers with intermediate braces. The effectiveness

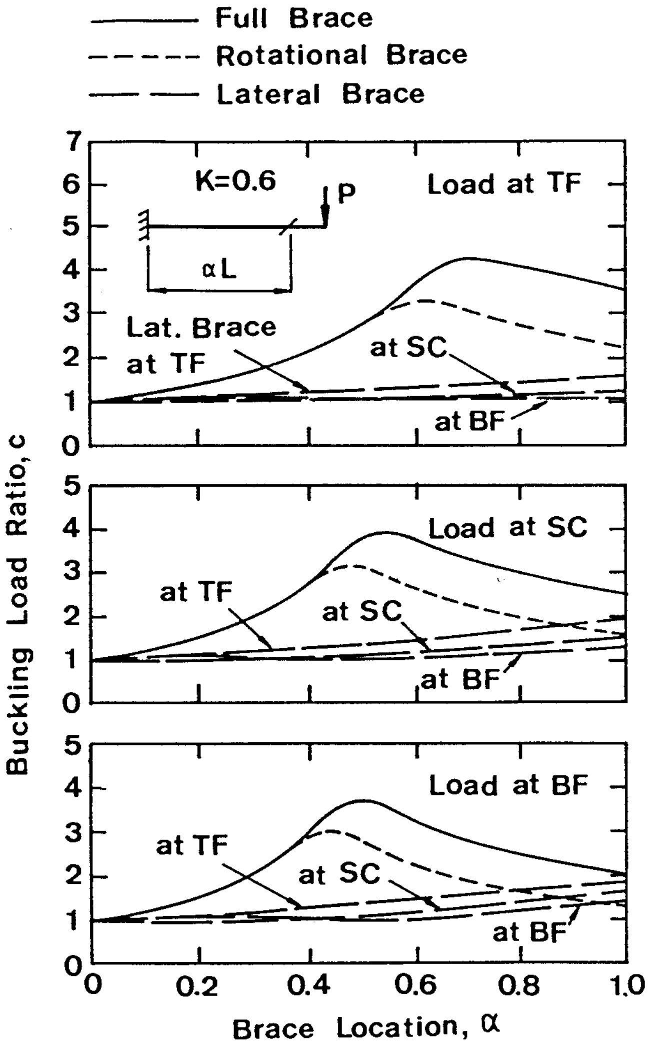


Fig. 33 — Comparison of Brace Types for Cantilevers with a Tip Load, K = 0.6.

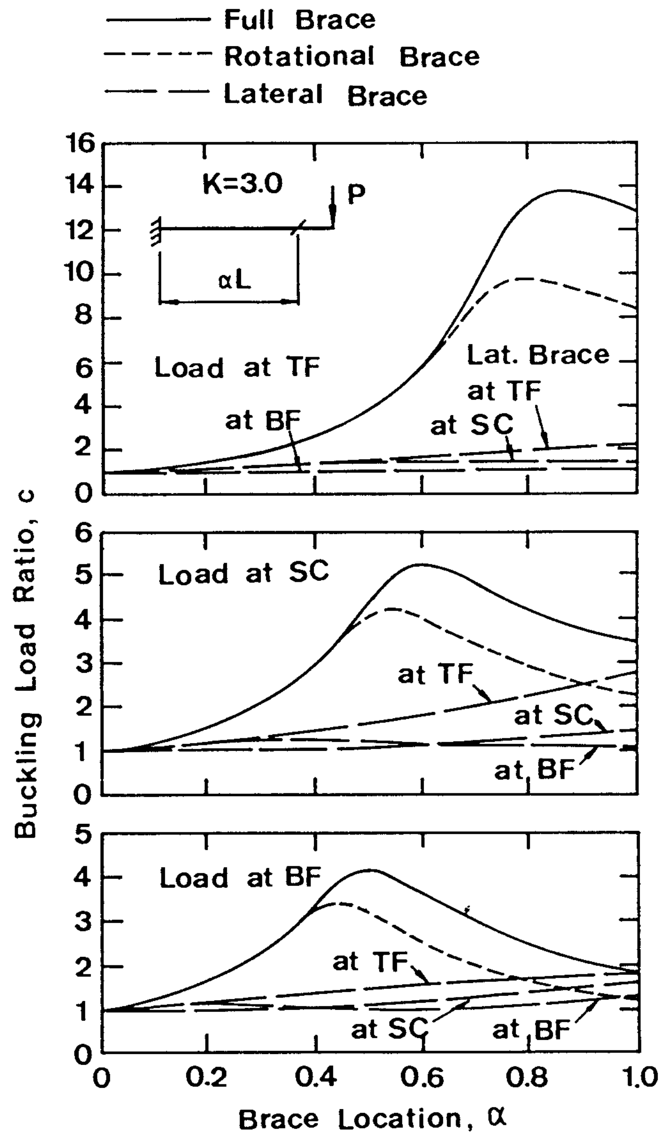


Fig. 34 — Comparison of Brace Types for Cantilevers with a Tip Load, K = 3.0.

of any type of brace on a beam depends on both K and the location of the brace.

For simply supported beams with a single lateral brace only, the brace is generally effective when acting above the shear centre. When the brace acts below the shear centre the effectiveness depends on the beam parameter K. In general the buckling resistance is significantly increased when loads act below the shear centre.

For cantilevers, the optimum location of a full brace for most cases varies between 0.4L to 0.7L from the fixed end support. For cantilevers with single lateral brace, the brace is best placed near the top (tension) flange level. However, this arrangement is not as effective as a rotational brace or a full brace.

4. Buckling of Laterally Continuous Beams

4.1 INTRODUCTION

This Section presents a simple, approximate method for determining the elastic buckling strength of laterally continuous beams. The term *laterally continuous* implies indeterminacy in the lateral (minor axis) direction imposed by braces both at and away from the beam ends. Typical examples of laterally continuous structures amenable to solution by this method are shown in Fig. 37. The basic requirements are:

(i) the structures are loaded by point loads at braced cross-sections;

(ii) the braces prevent lateral deflection and twist (full brace); and

(iii) elements of the structure are straight and major axis curvature effects are negligible.

The approximate analysis follows the principles of interaction buckling as set out in References 19 and 20 and for grids in Reference 21. A brief review of these principles is given in the following Section.

4.2 INTERACTION BUCKLING

In an interaction buckling analysis a laterally continuous beam is modelled as an assemblage of segments

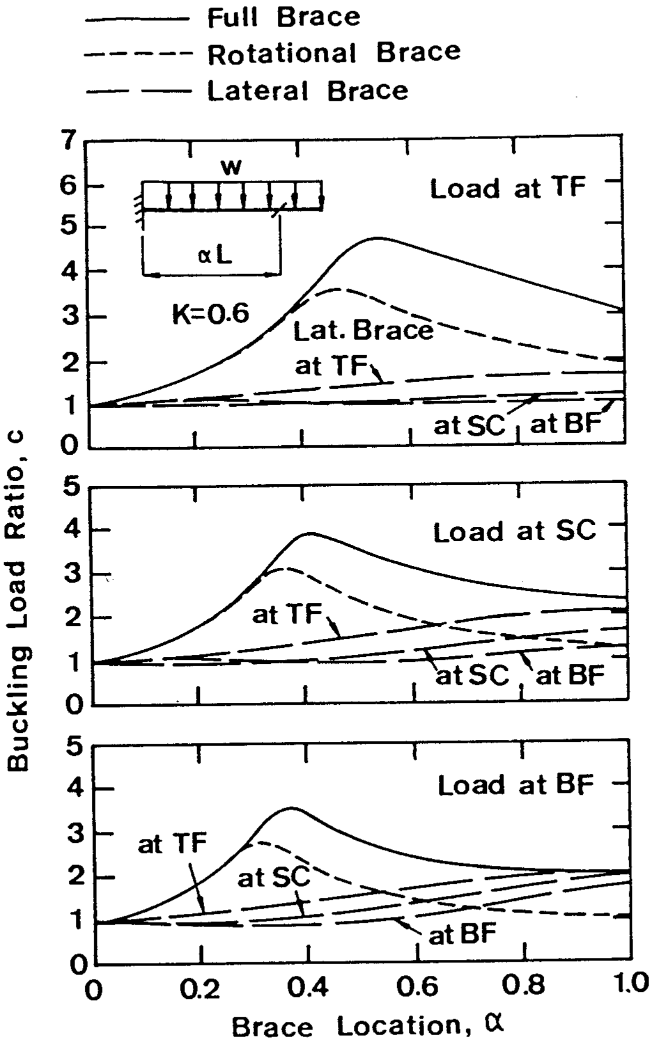


Fig. 35 — Comparison of Brace Types for Cantilevers with Uniformly Distributed Load, K = 0.6.

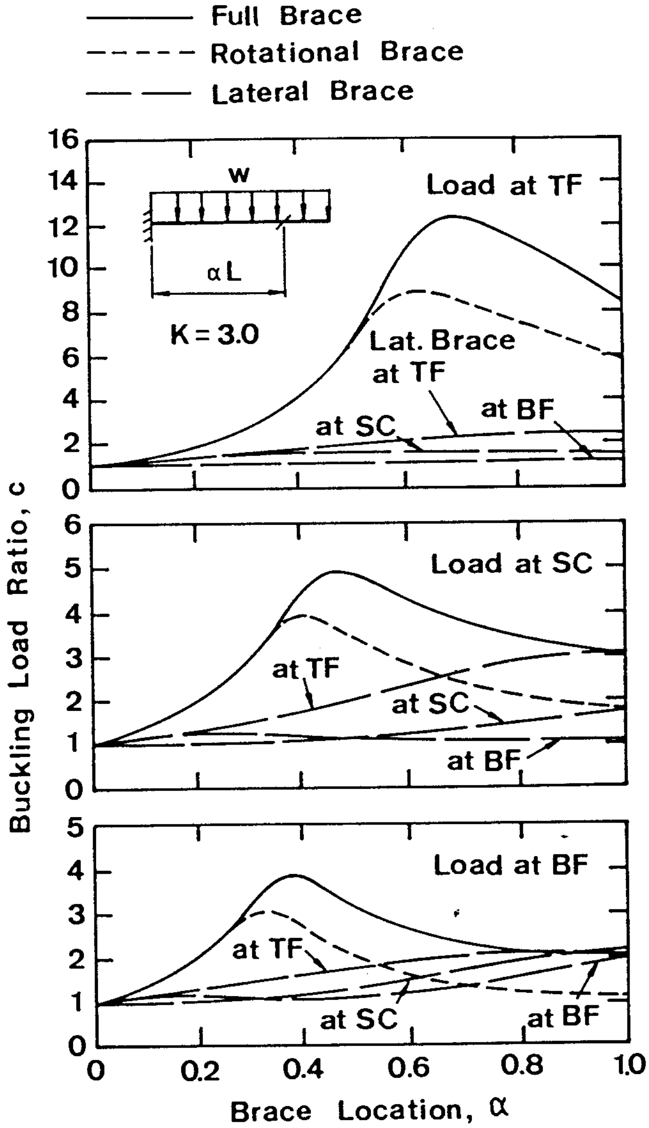


Fig. 36 — Comparison of Brace Types for Cantilevers with Uniformly Distributed Load, K = 3.0.

connected at braced points. If the beam is disturbed from its initially straight position it adopts patterns of minor axis displacement and twist consistent with the brace constraints of $u = \phi = 0$. Compatibility at braced points also requires minor axis bending and warping interaction between segments at their ends. Reference 20 shows that minor axis bending and warping stiffnesses at segment ends are major axis load-dependent and generally deteriorate parabolically with increasing load. While these stiffnesses are positive the segment, taken separately, is stable. If end stiffnesses become negative the segment requires end restraint. At loads less than the buckling load, some segments are able to offer minor axis bending and warping end restraint to more severely loaded segments and the beam as a whole is able to resist disturbance. At the buckling load the reserve of stiffness is zero. The beam enters a state of neutral equilibrium in which it is unable to resist a vanishingly small disturbance.

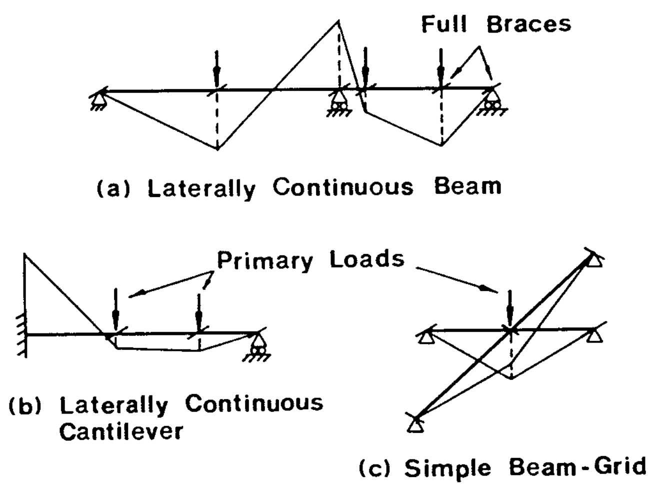


Fig. 37 — Laterally Continuous Structures.

4.3 MODEL FOR ANALYSIS

4.3.1 Critical Segment and Subassemblage

Buckling analysis of the isolated segments (e.g. with Equation (5)) can indicate whether a segment is likely to provide restraint or to require it as the beam approaches instability. Reference 19 introduced the term *critical segment* to identify that segment giving the lowest beam load factor for failure from an analysis excluding segment interaction. In the approximate method it is

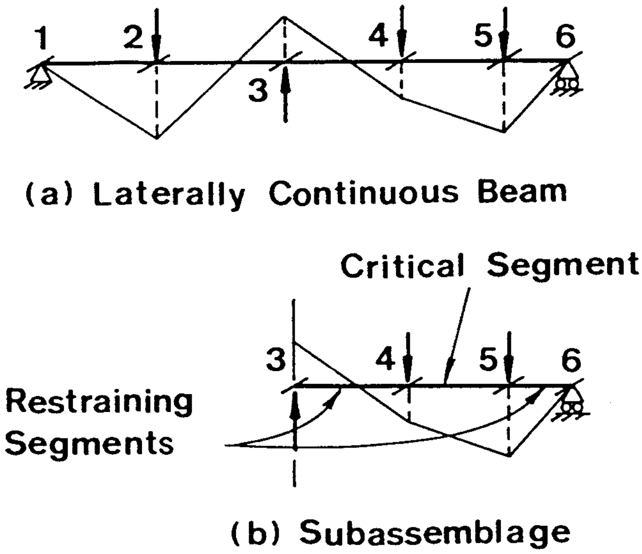


Fig. 38 — Typical Beam and Sub-assemblage.

assumed that the critical segment undergoes an earlier deterioration of stiffness and places a higher restraint demand than does any other segment. It therefore dominates in limiting beam capacity. An estimate of the restraint available to the critical segment can be made by examining the immediately adjacent segments. A sub-assemblage comprising the critical segment and the two adjacent segments is thus identified. It is assumed that the behaviour of the sub-assemblage adequately reflects that of the beam. A typical beam and sub-assemblage is shown in Fig. 38.

4.3.2 Boundary Conditions

Boundary conditions are required at the far ends of the restraining segments. Support conditions such as a simple support or fixed end are retained. If the far end continues on to another segment it is assumed that a restraint demand equal to that from the critical segment occurs at this end. Alternative conditions are discussed in Reference 20.

4.3.3 Restraint Stiffness

It is assumed that both the warping and minor axis bending stiffness at the end of a restraining segment vary in the manner given for the bending stiffness in Equation (9) (see Reference 20).

$$\text{Minor axis bending stiffness at segment end} = n\,[EI_y/L]_R\,[1 - [M/M_E]^2]_R \qquad (9)$$

where the subscript R refers to the restraining segment *and* $n = 3$ if the far end is simply supported, $n = 4$ if fixed, and $n = 2$ if continuous to another segment. The moment M_E is the elastic buckling moment of the restraining segment found from Equation (5) and M is the larger end moment in the segment. The variation in stiffness is parabolic and results in zero stiffness when $M/M_E = 1.0$. The restraint stiffness is described in a restraint parameter, G, where

$$G_{A,B} = \frac{2\left[\dfrac{EI_y}{L}\right]_C}{n\left[\dfrac{EI_y}{L}\right]_R\left[1-\left[\dfrac{M}{M_E}\right]^2\right]_R} \qquad (10)$$

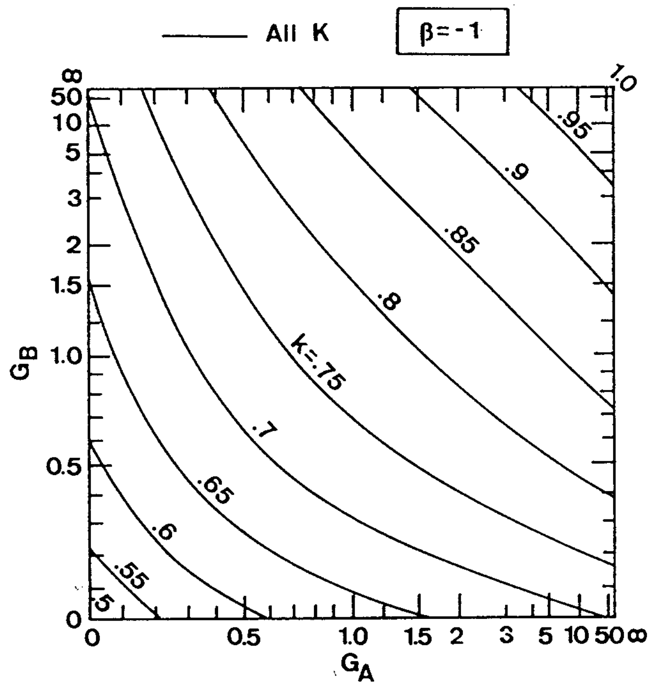


Fig. 39 — Effective Length Factor Chart for $\beta = -1$.

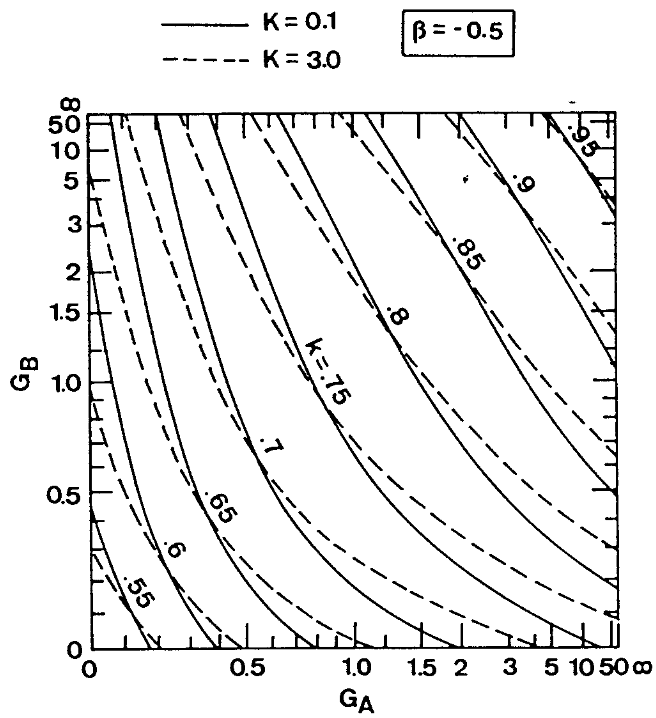


Fig. 40 — Effective Length Factor Chart for $\beta = -0.5$.

Subscript C refers to the critical segment and subscripts A and B refer to ends A and B of the critical segment (see Fig. 4 for A and B definition). The restraint parameter at an end describes equally the warping and minor axis bending stiffnesses available at that end.

4.4 EFFECTIVE LENGTH FACTORS

The load factor associated with the buckling of the restrained critical segment is assumed to approximate the load factor for beam failure.

The buckling moment of the critical segment can be expressed as:

$$M_F = m\,\pi/kL\sqrt{EI_yGJ}\sqrt{1+(K/k)^2} \qquad (11)$$

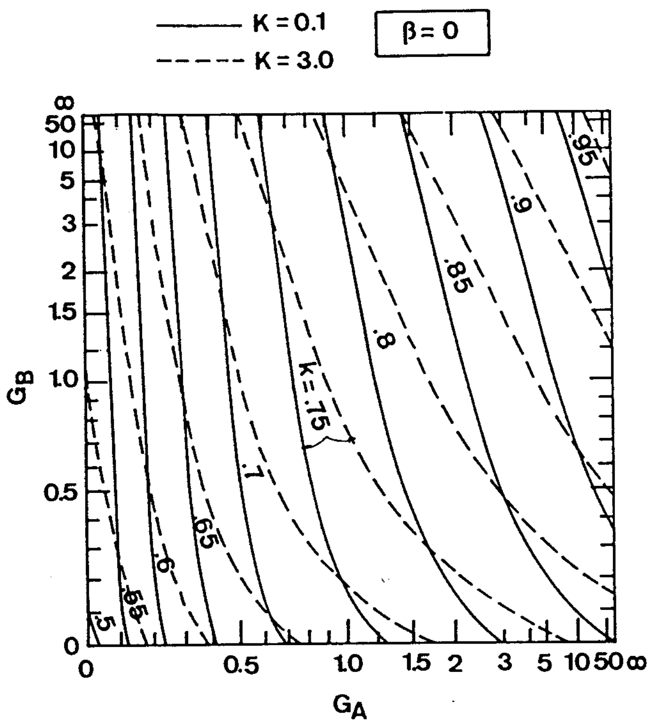


Fig. 41 — Effective Length Factor Chart for $\beta = 0$.

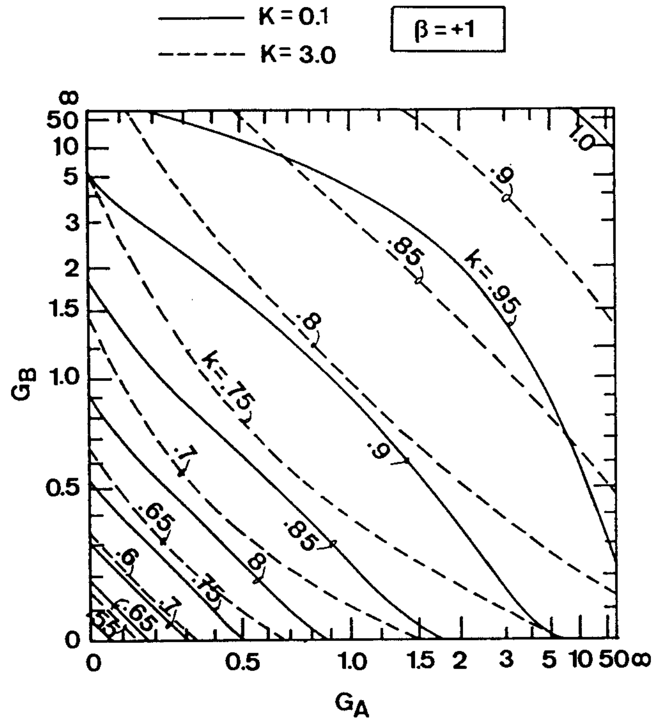


Fig. 43 — Effective Length Factor Chart for $\beta = +1$.

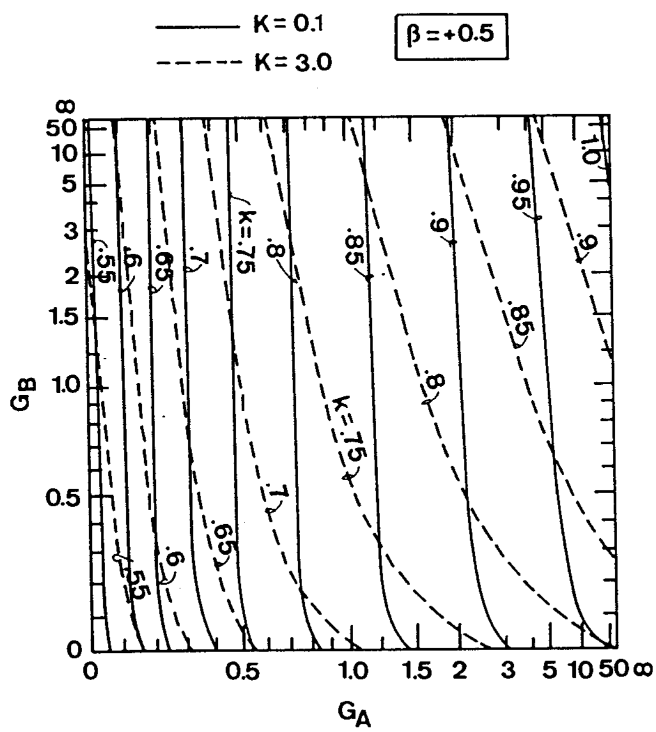


Fig. 42 — Effective Length Factor Chart for $\beta = +0.5$.

where k = effective length factor. When k = 1.0, Equation (11) reduces to Equation (5). Figs. 39 to 43 present effective length charts for restrained critical segments for the full range of β and for K varying from 0.1 to 3.0 (see References 20 and 22 for details and additional charts).

4.5 ANALYSIS PROCEDURE AND WORKED EXAMPLES

4.5.1 Summary of Steps

(1) Determine the major axis bending moment distribution.

(2) Determine K and β for each segment.

(3) For each segment calculate M_E from Equation (5) and the corresponding beam load factor, λ, to produce M_E. The segment with the lowest load factor, λ_C, is the critical segment. The two (at most) adjacent segments have higher load factors, λ_R.

(4) Assume a trial value of λ_F, the load factor at sub-assemblage buckling, and calculate G_A and G_B from Equation (10). Note that

$$\lambda_F/\lambda_R = [M/M_E]_R \qquad (12)$$

and this substitution can be made in Equation (10). The trial value λ_F should lie between λ_C and λ_R (min).

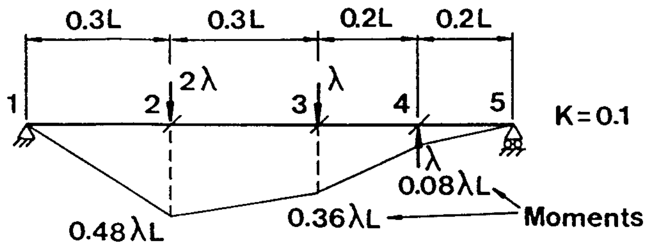
(5) Find the critical segment effective length factor, k, using the appropriate chart from Figs. 39 to 43, extrapolating linearly if necessary.

(6) Calculate the revised critical segment buckling moment, M_F, from Equation (11) and obtain a new load factor, λ_F (new). Note that

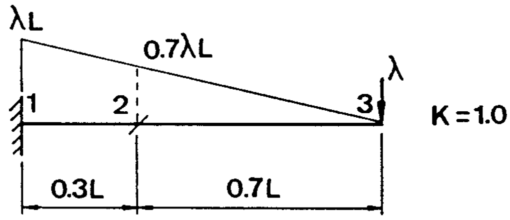
$$\lambda_{F(new)}/\lambda_C = [M_F/M_E]_C \qquad (13)$$

(7) Compare the new load factor, λ_F (new) with the assumed factor, λ_F (Step 4), and repeat Steps 4 to 6 if necessary until good agreement is obtained.

The process of cycling ensures consistency between assumed values at Step 4 and calculated values at Step 6. Usually only two or three cycles are required if a reasonable initial guess for λ_F is made at Step 4. Converging upper and lower bounds are found by choosing an initial value of λ_F equal to λ_C and subsequent values of λ_F equal to those calculated at Step 6.



(a) Laterally Continuous Beam



(b) Braced Cantilever

Fig. 44 — Worked Examples.

4.5.2 Worked Example 1

The analysis procedure is applied to the beam in Fig. 44a. Much of the data is summarized in Table 1.

TABLE 1: Analysis Data

Segments		1-2	2-3	3-4	4-5
β	=	0.0	−0.75	−0.22	0.0
K	=	0.33	0.33	0.5	0.5
$^*M_E$	=	19.3	12.46	26.9	30.7
$^{**}\lambda$	=	40.21	25.96	70.72	384.12

* all multiplied by $\sqrt{EI_yGJ}/L$
** all multiplied by $\sqrt{EI_yGJ}/L^2$

STEP 1 Calculate bending moments (see Fig. 44a).

STEP 2 Find β and K for each segment (see Table 1).

STEP 3 Calculate M_E and load factors, λ, to produce buckling in simply supported segments (see Table 1). Segment 2-3 is the critical segment. The sub-assemblage comprises segments 1-2, 2-3, 3-4.

STEP 4 Assume a value of λ_F and calculate G_A and G_B. End 2 of the critical segment has the higher end moment and is taken as end A. Usually a close guess can be made for λ_F from the information gathered at Step 3, but for the purposes of this example, an initial value of $25.96\sqrt{EI_yGJ}/L$ will be used.
Therefore

$$G_A = 2/3 \times 0.3L/0.3L \times \frac{1}{1 - \frac{25.96^2}{40.21^2}} = 1.14$$

and

$$G_B = 2/2 \times 0.2L/0.3L \times \frac{1}{1 - \frac{25.96^2}{70.72^2}} = 0.77$$

STEP 5 Find the critical segment effective length factor by interpolation between Figs. 39 and 40. Effective length factor, $k \approx 0.76$.

STEP 6 Calculate the revised critical segment buckling moment, M_F, and corresponding load factor, λ_F.

$$M_F = 1.13\pi/[0.3L \times 0.76]\sqrt{EI_yGJ}\sqrt{1 + [0.33/0.76]^2}$$
$$= 17.0\sqrt{EI_yGJ}/L$$

and

$$\lambda_F = 35.36\sqrt{EI_yGJ}/L^2$$

TABLE 2: Cycles of Analysis

Cycle	λ_F	G_A	G_B	k	λ_F(new)
1	25.96	1.14	0.77	0.76	35.36
2	35.36	2.94	0.89	0.82	32.40
3	32.40	1.90	0.84	0.79	33.80
4	33.80	2.27	0.86	0.81	33.00

Table 2 shows the convergence of λ_F which is taken as the mean value of the estimations at cycles 3 and 4, i.e.

$$\lambda_F = 33.4 \quad \sqrt{EI_yGJ}/L^2$$

A finite integral analysis gives

$$\lambda_F = 34.91 \quad \sqrt{EI_yGJ}/L^2$$

4.5.3 Worked Example 2

The analysis procedure can be refined to obtain better estimates of M_E for restraining segments at Step 3. This modification is advisable when the restraining segment at end A of the critical segment has a well restrained or fixed far end. Analysis of the cantilever in Fig. 44b illustrates this. Table 3 summarizes analysis data.

TABLE 3: Analysis Data

Segment	=	1-2	2-3
λ	=	-0.7	0.0
K	=	3.33	1.43
$^*M_E$	=	42.34	13.69
$^{**}\lambda$	=	42.34	19.56

* all multiplied by $\sqrt{EI_yGJ}/L$
** all multiplied by $\sqrt{EI_yGJ}/L^2$

STEP 1 Calculate bending moments (see Fig. 44b).

STEP 2 Find β and K for each segment (see Table 3).

STEP 3 Calculate M_E and load factors λ for buckling of simply supported segments (see Table 3). Segment 2-3 is critical and is restrained at end A.

STEP 4 (revised). Calculate M_F and load factor λ for buckling of segment 1-2 when simply supported at end 2 and fully restrained at end 1.

$$G_1 = G_A = 0.0$$
$$G_2 = G_B = \infty \quad \text{and } k = 0.68$$

Hence, the contents of Table 3 are revised to:

TABLE 4: Revised Data

Segment	=	1-2	2-3
$^*M_E$	=	89.4	13.69
$^{**}\lambda$	=	89.4	19.56

* all multiplied by $\sqrt{EI_yGJ}/L$
** all multiplied by $\sqrt{EI_yGJ}/L^2$

STEP 5 Estimate $\lambda_F \approx 38$

Therefore $G_A = 2/4 \times 0.3L/0.7L \dfrac{1}{1 - \left[\dfrac{38}{89.4}\right]^2} = 0.261$

$G_B = \infty$

STEP 6 Find the critical segment effective length factor from Fig. 41 ($\beta = 0$). k = 0.675

STEP 7 Calculate the revised critical segment buckling moment and corresponding load factor

$$M_F = 27.24\sqrt{EI_yGJ}/L$$

and

$$\lambda_F = 38.9\sqrt{EI_yGJ}/L^2 \text{ cf } 38\sqrt{EI_yGJ}/L^2$$

i.e. λ_F is approximately 38.45. The standard procedure leads to a value of 32.5 whereas a finite integral analysis gives

$$\lambda_F = 37.6\sqrt{EI_yGJ}/L^2$$

4.6 CONCLUSION

The approximate method produces accurate results for a wide range of laterally continuous structures as shown by comparison with rigorous solutions (finite element and finite integral) presented in References 20 and 21. Reference 19 outlines a simpler but less accurate method. Reference 23 develops an analysis procedure which accounts for additional warping and minor axis bending interaction which sometimes arises in beams with concentrated moment loading.

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STEEL CONSTRUCTION
LIST OF TITLES

This list gives the front cover titles for Vols. 16 to 19 inclusive. A list of titles for Vols. 1 to 15 appeared in Steel Construction Vol. 15 No. 4.

Vol. 16	No. 1	Portal Frame Design	L.J. Morris
	No. 2	Wide Span Structures	D.T. Wright R.G. Taylor
	No. 3	Tapered Masts	R.J. Veitch
	No. 4	Composite Columns	R.P. Johnson D.G.E. Smith
Vol. 17	No. 1	City Buildings	A. Firkins
	No. 2	Standard Connections	T.J. Hogan A. Firkins
		Two Long Span Structures	N. Long
	No. 3	Cold-Formed Columns	G.J. Hancock
	No. 4	Economical Steelwork	D.L.T Oakes
Vol. 18	No. 1	Composite Floors	R.Q. Bridge
	No. 2	Fire Protective Products	—
	No. 3	Curving Structural Steel	G. Riviezzi
	No. 4	Guyed Masts	ACADS Working Party
Vol. 19	No. 1	Tension Bracing	S.T. Woolcock S. Kitipornchai
	No. 2	Pin Connections	G. Riviezzi
	No. 3	Fire Testing Simulation — Thermal Response	I.D. Bennetts D.J. Proe I.R. Thomas
	No. 4	Fire Testing Simulation	